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FURTHER PROBLEMS IN THE THEORY AND DESIGN OF STRUCTURES.

FURTHER PROBLEMS

IN THE

THEORY AND DESIGN

OF

STRUCTURES

AN ADVANCED TEXT-BOOK

For the use of Students, Draughtsmen, and Engineers engaged in Constructional Work.

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WITH NUMEROUS ILLUSTRATIONS AND WORKED EXAMPLES.

Second Edition. Revised.
New Impression.

LONDON:
CHAPMAN & HALL, LTD.
1922.

PREFACE.

HAVING regard to recent developments in the subject and from experience in the lecture-room, the author has realised that the earlier book upon the *Theory and Design of Structures* omitted to deal with several problems that are of interest and importance to engineers. At the same time, the earlier book covers a fairly wide ground, which appears satisfactory for many engineers; and, after consideration, it was decided to keep the original book practically intact, and to write an additional volume.

The same general aim has been kept in view in the present as in the previous work: namely, to give treatments which are theoretically sound, while presenting those treatments in as clear and simple a manner as possible. An attempt has been made to give nearly all the steps involved in mathematical deductions; this is at the risk of criticism that the explanations are rather long, but experience has shown that many students find it very difficult to insert the gaps in mathematical reasonings that are usually made in text-books.

The first portion of the book deals with the method of Influence Lines, which has been developed to a considerable extent within recent years; next comes the Principle of Work and its application to deflections of framed structures, redundant frames, and rigid arches; finally, we have Portals and Wind Bracings and Secondary Stresses, the importance of the analysis of both having been more fully realised within recent years, though they have not yet received the adequate treatment which they deserve.

The author wishes to express his indebtedness to the various sources of information referred to in the text. Particular thanks are due to the Council of the Institution of Civil Engineers, Dr. F. C. Lea and Mr. Ralph Freeman, A.M.I.C.E., for permission to make use of their papers upon 'Influence Lines' and 'Two-hinged Spandril-braced Steel Arches' respectively; to the Editors of the *Architects' and Builders' Journal*, *Concrete and Constructional Engineering*, *Engineering Review*, and the *Engineering News* (New York) for matter appearing in these journals; and to Mr. F. L. Brown, for kindly reading the proofs.

The author will be glad to hear of any printer's errors or discrepancies that may be found.

EWART S. ANDREWS.

Goldsmiths' College, New Cross, S.E.
April, 1913.

PREFACE TO SECOND EDITION.

IN issuing the Second Edition of this book I would like to thank those readers who have been good enough to notify me of printer's errors. It is hoped that all such errors have now been corrected.

EWART S. ANDREWS.

204–6 *Bank Chambers.*
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June, 1920.

CONTENTS.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

FURTHER PROBLEMS IN THE THEORY AND DESIGN OF STRUCTURES.

The references in the text, such as A, p. 185, are to the author's 'Theory and Design of Structures.' Portions marked with an asterisk may *be omitted in the first reading.*

CHAPTER I.

INFLUENCE LINES.

THE determination of the stresses in a bridge when a rolling load crosses it forms an important part of the design of a bridge and necessitates a considerable amount of work. The method of influence lines, which we will now describe, considerably facilitates the work in some cases, and is therefore gradually coming into general use. The progress is, however, very gradual. The method was first outlined in Germany by Weyrauch in 1873, and was first noticed in the English language in 1887 in a paper by Professor Swain before the American Society of Civil Engineers; but it is only comparatively recently that it has figured in the engineering literature of this country.

DEFINITION.—*An influence line for any given point* P *on a structure is such a line that its ordinate at any point* Q *gives the bending moment, shear or similar quantity at* P *when a unit load is placed at* Q.

We shall limit our consideration to bending moment (B.M.) and shear and thrust influence lines.

There is a fundamental difference between the bending moment and shear influence lines and the ordinary diagrams for

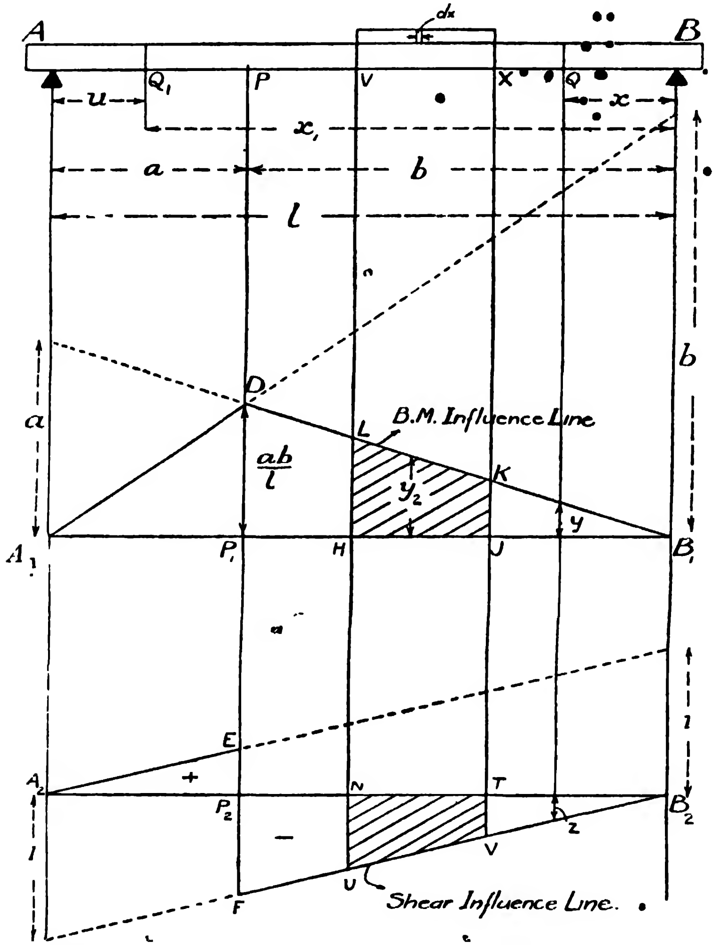


Fig. 1.—*Influence Lines for simply supported Beam.*

these quantities; this difference lies principally in the fact that the influence lines give at various points the values of the

quantities *at the same point* so that each point along the structure has its own influence lines.

SIMPLY SUPPORTED BEAMS.

We will proceed at once to the case of a simply supported beam to make this conception clear; we shall here obtain some rules that are true for any kind of beam whatever be its method of support, and will indicate such rules by the mark †.

If a *unit load* be placed at the point Q of a simply supported beam A B, Fig. 1, the bending moment at P

$$= M_P = R_A \cdot a$$

but $$R_A = \frac{1 \times x}{l} \text{ (by moments about B)},$$

$$\therefore\ M_P = \frac{1 \cdot x \cdot a}{l} = \frac{x\,a}{l}$$

(reckoning clockwise moments as positive).

This is proportional to x so that the B.M. influence line will be a straight line A_1 D B_1, the ordinate P_1 D being equal to $\frac{a\,b}{l}$, the reasoning for a point Q_1 on the other side of P being exactly similar.

To get the point D without calculation we will note that if A_1 D and B_1 D be produced to meet the reaction verticals, as shown in dotted lines, the intercepts will be equal to b and a respectively; by setting up from B_1 therefore a height equal to b and from A_1 a height equal to a, and joining across, we get the point D.

It is sometimes, however, more convenient to draw the influence line to an enlarged scale; in this case the intercepts will be set up to this enlarged scale.

Next consider the shear. (We shall consider upward shear to the right as positive, and to the left as negative.)

Shear at P

$$= S_P = -R_A = \frac{-1 \times x}{l}$$

This is proportional to x, so that the shear influence line between B and P is a negative straight line B_2 F.

If the load is at a point Q_1 at the other side of the point P,

$$S_P = -(R_A - 1)$$

$$= -\left(\frac{1 \times x_1}{l} - 1\right) = +\left(1 - \frac{x_1}{l}\right) = \left(\frac{l - x_1}{l}\right)$$

$$= \frac{u}{l}$$

The shear influence line between A and P is therefore a positive line A_2 E, which is drawn as shown.

Total B.M. and Shear due to a Number of Loads.—Suppose there are a number of loads W_1, W_2, W_3, &c., at points at which the ordinates of the B.M. and shear influence lines are y_1, y_2, y_3, &c., and z_1, z_2, z_3, &c.

Then total B.M. at P $= M_P = W_1 y_1 + W_2 y_2 + W_3 y_3 + \ldots$

$$= \Sigma W y$$

Total shear at P $= S_P = W_1 z_1 + W_2 z_2 + W_3 z_3 + \ldots$

$$= \Sigma W z$$

To get, therefore, the total bending moment or shear at the point from its influence lines due to a number of loads, we multiply each load by the ordinate of the influence line at the point where it acts and add together the results, due allowance being made for signs.†

Uniformly distributed Loads.—Let a portion V X of the beam carry a uniformly distributed load of p per unit length.

Then considering a short length $d\,x$, the B M. at P due to this length of load will be $p\,d\,x\,.\,y_2 = p \times$ area of corresponding strip of the influence line.

∴ Total B.M. at P due to uniform load

$$= p \times \text{area HJKL}$$

Similarly it follows that

Total shear at P due to uniform load

$$= p \times \text{area NTVU.}$$

To get, therefore, the total B.M. or shear at any point for a load uniformly distributed along the length of the beam, we multiply the load per unit length by the area of the B.M. or shear influence line under the portion covered.†

As an illustration of this rule take the case of a simply supported beam, and consider the central point of the span.

Fig. 2 shows the influence lines for this case. Suppose that we want to find the bending moment and shear at this point

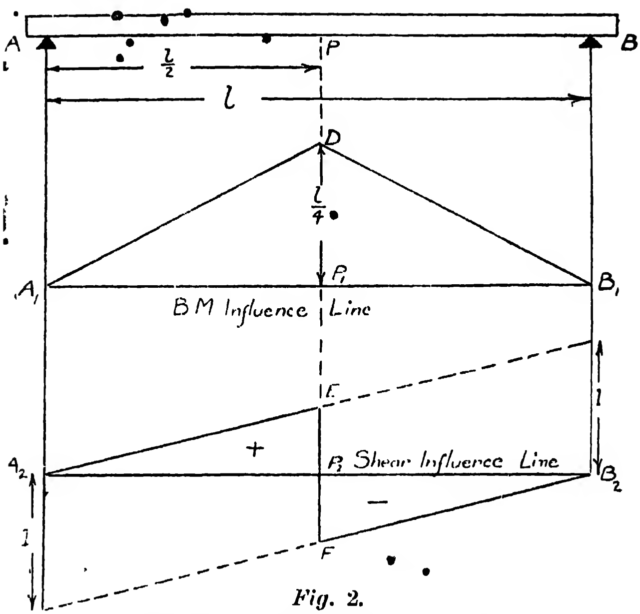


Fig. 2.
Influence Lines for Centre of Beam.

when the whole span is covered with a uniformly distributed load of intensity p.

The ordinate $P_1 D = \frac{l}{4}$ because $a = b = \frac{l}{2}$

$$\therefore \frac{ab}{l} = \frac{\frac{l}{2} \times \frac{l}{2}}{l} = \frac{l}{4}$$

B. M. at P when whole span is covered

$$= M_P = p \times \text{area of } \Delta\, A_1 D B_1$$
$$= p \times \tfrac{1}{2} A_1 B_1 . P_1 D$$

$$= p \times \frac{l}{2} \cdot \frac{l}{4}$$

$$= \frac{p\,l^2}{8}.$$

This is the well-known result.

Shear at P when whole span is covered

$$= S_P = \text{area of } \Delta\ A_2\,E\,P_2 - \text{area of } \Delta\ B_2\,F\,P_2 = 0$$

Next take the case where the load extends from the end A to the centre.

Then $\qquad M_P = p \times \text{area of } \Delta\ A_1\,P_1\,D$

$$= p \cdot \frac{1}{2} \cdot A_1\,P_1 \quad P_1\,D$$

$$= p \cdot \frac{l}{4} \cdot \frac{l}{4}$$

$$= \frac{p\,l^2}{16}$$

This agrees with the result obtained in the usual manner.

$$S_P = \text{area of } \Delta\ A_2\,P_2\,E$$

$$= \frac{1}{2} \cdot A_2\,P_2 \cdot P_2\,E$$

$$= \frac{l}{2} \cdot \frac{1}{2}$$

$$= \frac{l}{4}$$

This again checks with the ordinary method

MAXIMUM BENDING MOMENT AND SHEAR WITH LONG ROLLING LOADS.—Now take the case of a simply supported beam with a uniformly distributed load of length greater than the span.

Bending Moment.—It is clear from Fig. 1 that as the bending moment at the poi.. P for the uniform load V X is given by $p \times$ area H J K L, the bending moment will increase until H J becomes equal to $A_1\,B_1$; this is true from all positions of P so that we see that *the maximum bending moment at any point is obtained when the whole span is covered.*

This maximum bending moment

$$= p \times \text{area of } \Delta\ A_1\,D\,B_1$$

$$= p \times \frac{1}{2} A_1 B_1 . D P_1$$

$$= p \times \frac{l}{2} . \frac{a b}{l} = p . \frac{a b}{2}$$

$$= \frac{p a (l - a)}{2}$$

This is a parabolic relation; the parabola being of course the familiar one of height $= \frac{p l^2}{8}$.

Shear.—Next consider the shear at the point P as the load comes on from one side, say B. When the front of the load reaches the point V the shear at P is given by the area $-$ U N B_2; this area increases numerically until the point P is reached and then diminishes as the load passes farther on to the span owing to the area on the left-hand side of P_2 being positive. The maximum positive shear at P will clearly be equal to the area A_2 E P_2 and occurs when the back of the load passes the point P. We thus get the well-known rule that with a uniformly distributed load longer than the span, *the shear reaches a maximum when the front of the load reaches any given point and reaches another maximum of opposite sign when the end of the load leaves the point.*

MAXIMUM BENDING MOMENT AND SHEAR WITH SHORT UNIFORMLY DISTRIBUTED ROLLING LOAD.—*Bending Moment.*—If the uniformly distributed rolling load be of length l', Fig. 3, less than the span, then to find the maximum bending moment that occurs at the point P, we have to find the position of H J to make the area H L D K J a maximum; the position will clearly be somewhat as shown in the figure, because if we make $P_1 J' = H J$, which would correspond to the part of the load being at the point P (this obviously giving the maximum bending moment for the load anywhere on the right of P), it is clear that the area J K K′ J′ will come less than the area H L D P_1.

Now area H L D K J

$$= \text{area } H L D P_1 + \text{area } P_1 D K J$$

$$= H P_1 \left(\frac{L H + D P_1}{2} \right) + P_1 J \left(\frac{D P_1 + K J}{2} \right)$$

$$= x\left\{\frac{(a-x)\tan\alpha + a\tan\alpha}{2}\right\} + x'\left\{\frac{b\tan\beta + (b-x')\tan\beta}{2}\right\}$$

$$= x\left(a\tan\alpha - \frac{x\tan\alpha}{2}\right) + x^1\left\{b\tan\beta - \frac{x^2\tan\beta}{2}\right\} \dots (1)$$

Now $h = a\tan\alpha = b\tan\beta$

$$\therefore \text{ above expression} = x\left(h - \frac{xh}{2a}\right) + x'\left\{h - \frac{x'h}{2b}\right\}$$

$$= h\left\{x - \frac{x^2}{2a} + x' - \frac{x'^2}{2b}\right\}$$

$$= h\left\{x - \frac{x^2}{2a} + (l'-x) - \frac{(l'-x)^2}{2b}\right\}$$

$$= h\left\{l' - \frac{x^2}{2a} - \frac{(l'-x)^2}{2b}\right\} \dots\dots\dots (2)$$

This will be a maximum when $\frac{dA}{dx} = 0$

$$i.e. \quad -\frac{2x}{2a} - \frac{2(l'-x)(-1)}{2b} = 0$$

$$i.e. \quad \frac{x}{a} = \frac{l'-x}{b}$$

$$\text{or} \quad \frac{x}{l'-x} = \frac{a}{b}$$

$$i.e. \quad \frac{x}{l'} = \frac{a}{a+b} = \frac{a}{l}$$

The maximum bending moment at any point occurs therefore when the point divides the load in the same ratio as it divides the span. Putting these results into equation (2) therefore we get:

Max B.M. at P

$$= ph\left\{l' - \frac{x^2}{2a} - \frac{b^2x^2}{a^2 \cdot 2b}\right\}$$

$$= ph\left\{l' - \frac{al'^2}{2l^2} - \frac{ba^2l'^2}{a^2 \cdot 2l^2}\right\}$$

$$= phl'\left\{1 - \frac{al'}{2l^2} - \frac{bl'}{2l^2}\right\}$$

$$= phl'\left\{1 - \frac{l'}{2l^2}(a+b)\right\}$$

$$= p h l' \left\{ 1 - \frac{l' l}{2 l^2} \right\}$$

$$= p h l' \left\{ 1 - \frac{l'}{2 l} \right\} \qquad \cdots (3)$$

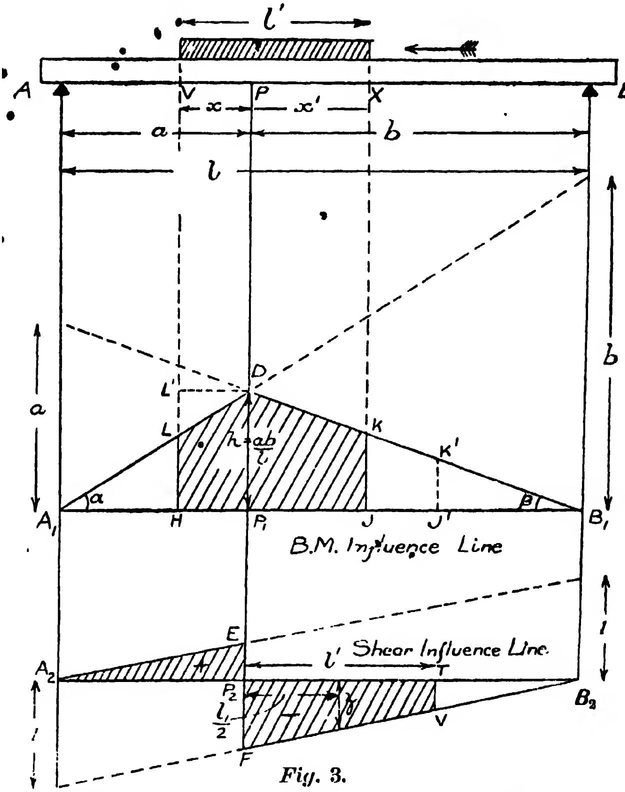


Fig. 3.

Influence Lines for uniformly distributed Loading.

If the total load is W (pl')

This comes Max. B. M. $= W h \left(1 - \dfrac{l'}{2 l} \right)$

a the centre $h = \dfrac{l}{4}$

$$\text{Then Max. B. M.} = \frac{W\,l}{4}\left(1 - \frac{l'}{2\,l}\right)$$

$$= \frac{W\,l}{4} - \frac{W\,l'}{8} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

This agrees with the result obtained by the ordinary method. [A, p. 185.]

Shear.—Turning to the shear influence line, it will be clear that as the load travels on from the end B, the negative shear at P increases until the front of the load reaches P; the shear is then equal to

$$S_P = -p \times \text{area } F P_2 T V$$

$$= -p \,.\, l'\left\{\frac{P_2 F + T V}{2}\right\}$$

$$= \frac{-W}{2}\left\{\frac{b}{l} + \frac{b - l'}{l}\right\}$$

$$= \frac{-W}{2}\left\{\frac{2b - l'}{l}\right\}$$

$$= \frac{-W}{l}\left(b - \frac{l'}{2}\right)$$

$$= -W z$$

$$\left(\text{because } \frac{z}{1} = \frac{b - \frac{l'}{2}}{l}\right).$$

The maximum positive shear will clearly occur when the end of the load leaves P and will be equal to p × area $\Delta\, A_2 P_2 E$ if, as in the case shown in the figure $l' > a$.

Loads applied at definite Points.—In many plate girder bridges the loads are transferred by the floor system o stringers and cross-girders at definite load points. We shall now prove that the influence line between such load points will always be straight.

Take the case of a load W, Fig. 4, applied at a point Q and transmitted at points C, D, the ordinates of an influence line of which are y_C and y_D.

Then load transmitted at D $= \dfrac{W x}{d}$, and load transmitted at C $= W\left(1 - \dfrac{x}{d}\right)$.

$$\therefore \text{ shear or B.M. at P} = \frac{Wx}{d} \cdot y_D + W\left(1 - \frac{x}{d}\right) \cdot y_C$$

$$= W\left\{(y_D - y_C)\frac{x}{d} + y_C\right\}$$

This is clearly a linear relation, so that the influence line between C and D will be a straight line.

We can therefore always determine the influence line for a

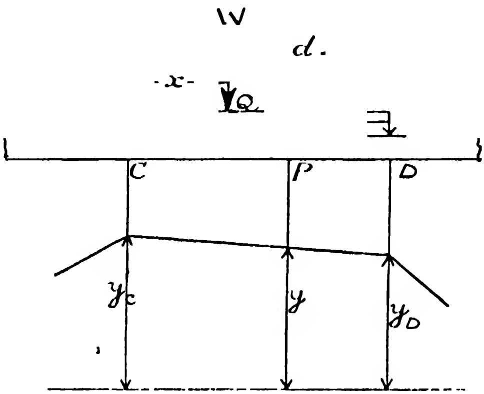


Fig. 4.

structure loaded at definite points by calculating the ordinate at each of the load points and joining up by straight lines.†

Irregular Load Systems.—The above cases enable us to deal with the design of many bridges, because the loads are very often given as equivalent uniform loads. In many cases, however, the loads are specified as wheel loads at certain distances apart, so that it is necessary to find the maximum B.M. and shear for this irregular load system. This can be done by trial by advancing the load gradually on to the bridge and finding the B.M. curve for each position; but this is a tedious process, and the work is greatly facilitated by the following consideration of influence lines:—

POSITION OF LOADS TO PRODUCE MAXIMUM B.M. AT A GIVEN POINT.—Let the loads be such that the resultant load on the

right of the given point P, acting through the centre of gravity is W_1 (Fig. 5), and the resultant on the left is W_2.

Then total B.M. at P $= M_P = W_1 y_1 + W_2 y_2$.

This will be clearly seen by considering one set of the loads, say W_2, separately. Due to this load, $M_P = R_B . b$, and R_B is clearly the same for W_2 as for the separate loads.

To find the position of the load to give the maximum B.M. at

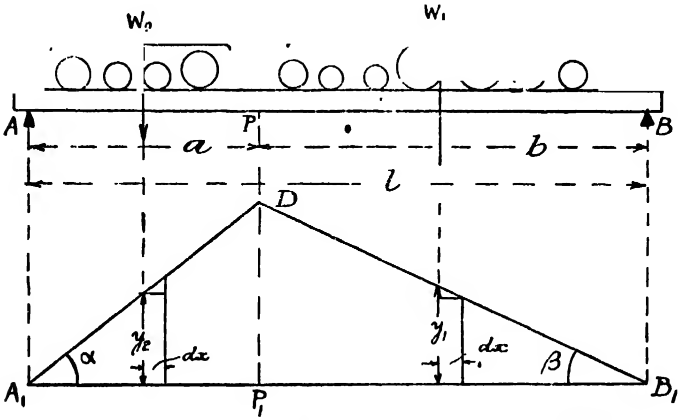


Fig. 5.—Maximum B.M. for Isolated Load System.

P we will move the loads a short distance dx and see how this affects the B.M.

$$y_2 \text{ becomes } y_2 + dx \,.\, \tan\alpha$$

$$y_1 \text{ becomes } y_1 - dx \tan\beta$$

$\therefore$ new value of $M_P = W_1 (y_1 - dx \tan\beta) + W_2 (y_2 + dx \tan\alpha)$

$\therefore$ increase of B.M. $= d\,M_P = W_2 \tan\alpha\, dx - W_1 \tan\beta\, dx$

$$\therefore \frac{d\,M_P}{dx} = W_2 \tan\alpha - W_1 \tan\beta$$

$\therefore$ M_P is a maximum when $W_2 \tan\alpha - W_1 \tan\beta$ changes sign.*

* We cannot in this case take for the determination of our maximum the condition that $\frac{d\,M_P}{dx} = 0$ because for an isolated load system $\frac{d\,M_P}{dx}$ will have sudden steps and there will probably be no value which comes exactly zero.

$$\therefore \frac{W_2 P_1 D}{a} - \frac{W_2 . P_1 D}{b} \text{ must change sign.}$$

i.e., $\frac{W_2}{a} - \frac{W_1}{b}$ must change sign.

The values of W_2 and W_1 can change only when a load passes A, P or B.

If a load passes A in the direction of movement that we are considering, W_2 is increased, and if a load passes B, W_1 is decreased, and neither of these changes can make $\frac{W_2}{a} - \frac{W_1}{b}$ change sign; but if a load passes P, W_1 increases and W_2 decreases, so that $\frac{W_2}{a} - \frac{W_1}{b}$ can change sign only by a load passing P, therefore *to get a maximum B.M. a load should be placed at* P, *so that if the load is considered as part of* W_2 *the expression* $\left(\frac{W_2}{a} - \frac{W_1}{b}\right)$ *will be positive, but if considered as part of* W_1, *the expression* $\left(\frac{W_2}{a} - \frac{W_1}{b}\right)$ *will be negative.*

With continuous loading the above result gives a simple rule :—

Then $$\frac{W_2}{a} - \frac{W_1}{b} = 0$$

i.e. $$\frac{W_2}{W_1} = \frac{a}{b}$$

i.e. $$\frac{W_2}{W_1 + W_2} = \frac{a}{a + b} = \frac{a}{l}$$

i.e. $$\frac{W_2}{a} = \frac{\text{total load}}{l}$$

or in words :—*The average load per unit length on one side of the point must equal the average load on the whole span.*

This agrees with the result obtained on p. 8.

NUMERICAL EXAMPLE.—*Take for example the load system given in Fig. 6, and find the maximum B.M. at the centre of a 40 ft. span.* In this case $a = b = 20$ ft.

Now as a trial place the second load at the centre and apply the test given above.

If the load is considered part of W_2,

$$\frac{W_2}{a} - \frac{W_1}{b} = \frac{80}{20} - \frac{10}{20} = 3\cdot5$$

If part of W_1,

$$\frac{W_2}{a} - \frac{W_1}{b} = \frac{60}{20} - \frac{30}{20} = 1\cdot5$$

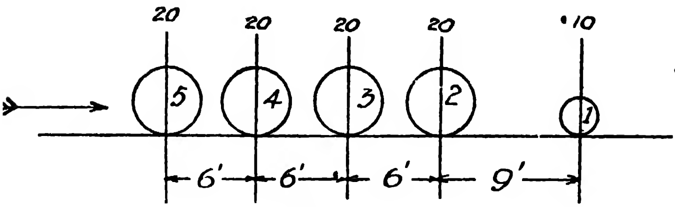


Fig. 6.

This does not change sign, so that this position will not give a maximum.

Next put the third load at P.

Taking this load as part of W_2,

$$\frac{W_2}{a} - \frac{W_1}{b} = \frac{60}{20} - \frac{30}{20} = 1\cdot5$$

As part of W_1,

$$\frac{W_2}{a} - \frac{W_1}{b} = \frac{40}{20} - \frac{50}{20} = -\cdot5$$

This changes sign so that this gives the position required.

$$\text{For this position } R_A = \frac{10 \times 5 + 20\,(14 + 20 + 26 + 32)}{40}$$

$$= 1\cdot25 + 46 = 47\cdot25 \text{ tons.}$$

$$\therefore\ M_P = 47\cdot25 \times 20 - 20\,(6 + 12)$$

$$= 20\,(47\cdot25 - 18) = 585 \text{ ft.-tons.}$$

Position of Loads for Maximum Shear at any Point.—Let the beam be loaded with a series of loads, Fig. 7. It will be clear from the influence line that loads on the right cause negative shear, while loads on the left cause positive shear. If the whole series be moved to the right, the positive shears will be increased numerically until one of the loads passes the point P; a maximum shear occurs at each point, therefore, whenever a load reaches the point, but it is the maximum of these maxima that we require to

find. It is clear that this maximum will occur when there are few or no loads beyond P, because the loads beyond P cause negative shear.

Let two consecutive loads w_1 and w_2 be at distance c apart, and let them in turn be brought just up to the point P.

Let W_1 be the resultant load of the loads beyond P and including w_1, and let W_2 be the resultant load before P.

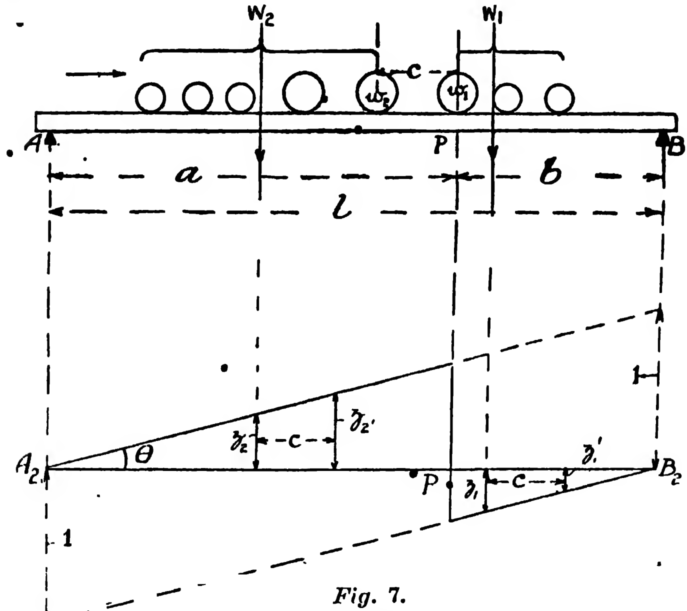


Fig. 7.

Maximum Shear for Isolated Load System.

Directly the load w_1 passes P, the shear is suddenly diminished by an amount w_1, the shear then becoming $S_P = W_2 z_2 - W_1 z_1$.

As the load w_2 approaches P the shear increases until when it is just in front of P the shear becomes $S^1_P = W_2 z_2^1 - W_1 z_1^1$.

$$\begin{aligned} \text{The increase} &= W_2 (z_2^1 - z_2) - W_1 (z_1 - z_1^1) \\ &= W_2 \,.\, c \tan \theta - W_1 (- c \tan \theta) \\ &= (W_2 + W_1) c \tan \theta \end{aligned}$$

$$= \frac{Wc \times 1}{l} = \frac{Wc}{l}$$

where W = total load on the span.

∴ Net increase in shear due to the movement $= \frac{Wc}{l} - w_1$.

If, therefore, this expression comes positive, the shear in the second position will be greater than that in the first.

If all the loads should be equal and equally spaced, then $\frac{Wc}{l}$ will always be less than w_1, so that the maximum shear occurs when the first load reaches the point.

If, however, the first load is small or the first space c is large, then it may easily happen that the second or even the third load should be placed at the point to give the maximum shear.

It should be noted that if any load moves off the span in the movement indicated above, such load should not be included in the value of W, but any additional load that comes on may be included; if, however, the inclusion of this additional load changes the sign of the above expression, the actual shears for the two positions should be calculated.

NUMERICAL EXAMPLE OF SHEARS.—*Take as before the loading and span indicated in Fig.* 8, *and find the maximum shear at the right-hand end* B *and at* 10 *feet from* A.

Let the first load be placed at B, and then let the second load be placed at B.

$$\frac{Wc}{l} = \frac{90 \times 9}{40} = 21{\cdot}5$$

$$w_1 = 10$$

$$\therefore \frac{Wc}{l} - w_1 \text{ is positive.}$$

∴ Shear is greater with load 2 at B than with load 1 at B.

Now bring the next load up to B.

$$\frac{Wc}{l} = \frac{80 \times 6}{40} = 12$$

$$w_1 = 20$$

$$\therefore \frac{Wc}{l} - w_1 \text{ is negative.}$$

∴ Shear is greater with load 2 at B than with load 3 at B.

∴ *Maximum shear at* B *occurs when load* 2 *is at* B.

Then shear

$$R_B = 20\frac{(22 + 28 + 34)}{40} + 20 = 62 \text{ tons.}$$

Now take a point C at 10 feet from A. Let the first load be placed at this point. Then neglecting the load that comes on

$$\frac{Wc}{l} = \frac{30 \times 9}{40} = 6{\cdot}75$$

$$w_1 = 10$$

$$\therefore \frac{Wc}{l} - w_1 \text{ s negative.}$$

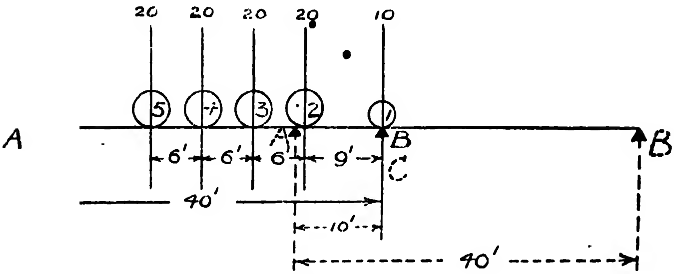


Fig. 8.

Including the load we get

$$\frac{Wc}{l} = \frac{50 \times 9}{40} = 11{\cdot}25$$

$$w_1 = 10$$

$$\therefore \frac{Wc}{l} - w_1 \text{ is positive.}$$

In this case, therefore, the inclusion of the additional load changes the sign of the expression so that we must find the shears.

With 1st load at point C

$$R_B = \frac{20 \times 1 + 10 \times 10}{40} = 3 \text{ tons.}$$

$$\therefore \text{Shear at C} = R_B = 3 \text{ tons.}$$

With 2nd load at point C

$$R_B = \frac{20 \times 4 + 20 \times 10 + 10 \times 19}{40} = 11{\cdot}75 \text{ tons.}$$

$$\text{Shear at C} = R_B - 9 = 2{\cdot}75 \text{ tons.}$$

$\therefore$ *Maximum shear at* C *occurs when load* 1 *is at* C.

Procedure to facilitate Calculations.—The following method was suggested by Mr. F. C. Lea, D.Sc., A.M.I.C.E., in a paper in Vol. CLXI. of the *Proc. Inst. C.E.* This paper was, we believe, the first British publication on the subject of influence lines, and has been consulted in writing the present volume.

Choose first a convenient bending moment scale, say 1″ = 50 foot-tons. We have seen that the ordinate of the B.M. influence line at the point under consideration is given by

$$\text{PD} = \frac{a\,b}{l}$$

$$\textit{i.e.}\ \text{PD} = \frac{a\,(l - a)}{l} = \frac{a\,l - a^2}{l}$$

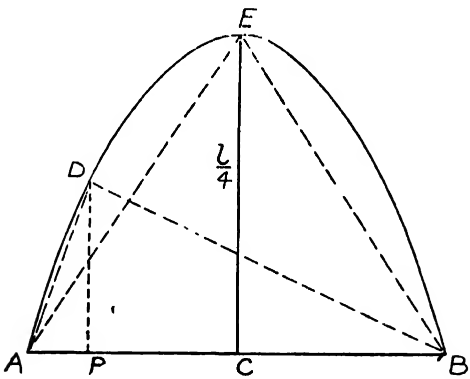


Fig. 9.

It is clear from this that the locus of the point D is a parabola, the maximum ordinate of which occurs for $a = \frac{l}{2}$ and is equal to $\dfrac{\frac{l^2}{2} - \frac{l^2}{4}}{l}$ *i.e.* $= \frac{l}{4}$

For any span A B, Fig. 9, draw this parabola of height $\frac{l}{4}$ to a scale say 12 inches = 50 feet. Now on squared paper take a length $C_1\,E_1$, Fig. 10, 12 inches long, set up the loads W_1, W_2, &c.,

to a scale of, say, 1 inch to 1 ton each starting from E_1, and draw radiating lines from C to the points marked on E_1 F, marking

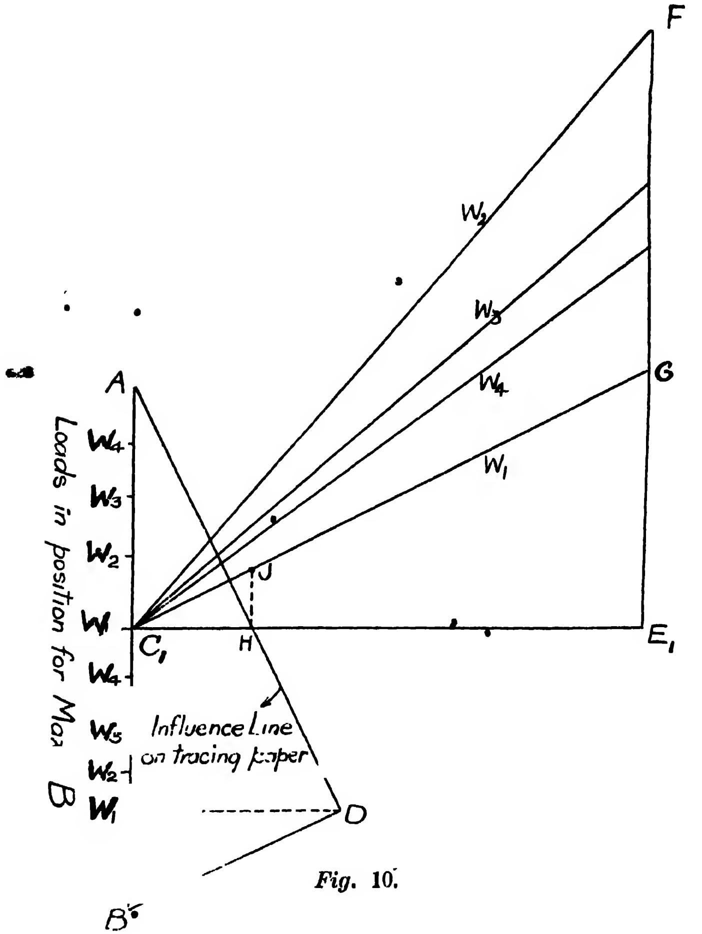


Fig. 10.

these lines W_1, W_2, &c., as shown. Now on tracing paper draw the influence line A D B for any point P along the span, this line being obtained from the parabola of Fig. 9; on the tracing paper draw the positions of the loads to give the maximum bending moment

—this position having been found beforehand by the method previously explained.

Slide the tracing paper along until W_1 is at C_1; the influence line cuts C_1 E_1 in H; then the ordinate J H measured on the scale 1″ = 50 foot-tons will be the bending moment in foot-tons due to the load W_1. The tracing paper is now moved until W_2 is at C_1 and the bending moment is then read off up to the line W_2 and so on, the total bending moment being obtained by adding together the various parts.

Proof.

$$\frac{J\,H}{C_1\,H} = \frac{G\,E_1}{C\,E_1} = \frac{W_1}{C\,E_1}$$

$$\therefore \quad J\,H = \frac{W_1 \times C_1\,H}{C_1\,E_1}$$

Now bending moment at P due to W_1

$$= {}_1M_P = W_1 \times C_1\,H$$
$$= J\,H \times C_1\,E_1$$

If, therefore, C_1 E_1 = 1 foot, and the parabola is drawn to a scale 12″ = 50 feet, J H measured to the scale 1″ = 50 foot-tons will give the bending moment due to the load W_1.

Use of Centre of Gravity of Locomotives, &c.—In using the influence line method, where the load consists of several locomotives, it will often be found useful to mark on the load strip the centre of gravity of each set of loads. If, then, the whole of an engine is under the line of the influence diagram, it will be necessary only to scale off the ordinate under the centre of gravity to get the bending moment for the whole engine.

CHAPTER II.

INFLUENCE LINES FOR SIMPLY SUPPORTED FRAMES OR TRUSSES.

THE influence line method of dealing with rolling loads is particularly applicable to frames or trusses because the stresses in such trusses can be found by a consideration of shears or moments.

In nearly all well-designed framed structures the loads are applied at the nodes or joints only so that we have here the case where loads are applied at definite points.

Parallel Flange Trusses.—Warren Girder.—Now take the particular case of the Warren girder (Fig. 11).

B.M. INFLUENCE LINE.—As will be readily understood, if we want the stress in C D we take moments about P, and so we require the B.M. influence line for the point P.

Take the unit load at any point Q_1 between D and B. Then

$$M_P = R_A \cdot a = \frac{1 \cdot u}{l} \cdot a = \frac{u\,a}{l}$$

This is a linear relation as before, therefore between D and B the B.M. influence line is $B_1 E$, where

$$D_1 E = y_d = \frac{a \cdot DB}{l} = \frac{a(b + c - d)}{l} = \frac{a\,b}{l} + \frac{a(c - d)}{l}$$

Similarly between A and C the B.M. influence line is $A_1 F$ where

$$C_1 F = y'_c = \frac{b \cdot (a - c)}{l} = \frac{a\,b}{l} - \frac{b\,c}{l}$$

From what we have proved on p. 11 it follows that the B.M influence line is straight between C and D, so that the influence line is completed by joining E F.

Now the portions $B_1 E$ and $A_1 F$ of the influence line are, from the above reasoning, the same as the corresponding portions of the influence line for a point P on a simply supported beam of

span A B. It follows, therefore, that if these lines are produced, they will intersect at H vertically below P, and the ordinate at H will equal $\frac{a\,b}{l}$. This gives us the following simple method of drawing the B.M. influence line for P :—

Set up $P_1\,H = \frac{a\,b}{l}$ and join $H\,A_1$, $H\,B_1$ intersecting $C\,C_1$ and

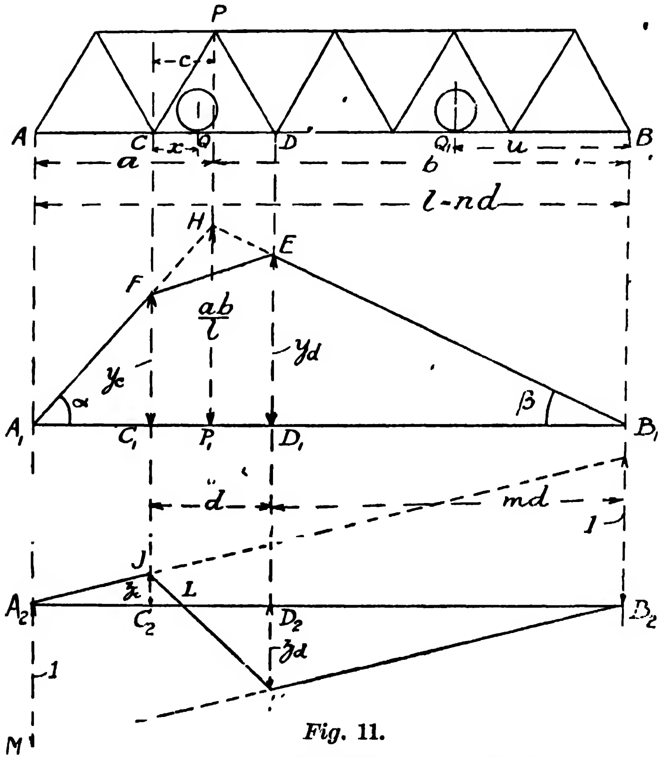


Fig. 11.

Influence Lines for Warren Girder.

$D\,D_1$ in F and E, and join F E or draw A_1 F and B_1 E by the construction explained with reference to Fig. 1.

Then $A_1\,F\,E\,B_1$ = B.M. influence line.

In finding the stresses in the upper flanges we shall have to

take moments about points such as C and D, and for these the influence lines will be the same as for simply supported beams.

SHEAR INFLUENCE LINE.—To get the stresses in the diagonals C P, P D, we require, it will be remembered, to find the shear in the panel C D. If a load be placed between B and D, the shear in this panel $= - R_A$.

The shear influence line between B and D will therefore be a straight line B_2 K obtained as before explained, and by similar reasoning it follows that between A and C will be the straight line A_2 J; then since the influence line between the load points must be a straight line, the influence line is completed by joining J K.

SINGLE LOAD AND UNIFORMLY DISTRIBUTED LOAD.—*Position of Load for Maximum B.M.*—It is clear from the B.M. influence line that with a single load the value of M_P is a maximum when the load reaches D, and that with a uniformly distributed load of length not less than the span, the maximum value of M_P occurs when the whole span is covered, since M_P in this case varies as the area of the corresponding portion of the influence line diagram.

Position of Loads for Maximum Shear.—To get a maximum positive shear with a single load it is clear that the load must be placed at C, while for a maximum negative shear it must be placed at D.

If the load is uniformly distributed, and of length not less than L B_2, the maximum shear will be equal to the area L K B_2, and will occur when the front of the load reaches L. Now

$$\frac{L\,D_2}{L\,C_2} = \frac{z_d}{z_c} = \frac{B_2\,D_2}{A_2\,C_2}$$

$$\frac{L\,D_2}{L\,C_2 + L\,D_2} = \frac{B_2\,D_2}{A_2\,C_2 + B_2\,D_2}$$

$$= \frac{m\,d}{A_2\,B_2 - C_2\,D_2} = \frac{m\,d}{(n-1)\,d}$$

$$\therefore \frac{L\,D_2}{d} = \frac{m}{n-1}$$

$$L\,D_2 = \left(\frac{m}{n-1}\right)d$$

Therefore: *The maximum shear in the $(m + 1)^{th}$ panel of a girder of n panels occurs for continuous loading when $\left(\frac{m}{n-1}\right)$ of the panel is covered.**

Irregular Load System.—In the case of an irregular load system we proceed to find the test for maximum stresses in a manner similar to that employed for simply supported beams.

Maximum Bending Moment.—When moments are taken about the lower flange nodes such as C, D, the condition for maximum moment will be the same as was proved on p. 13 for simply supported beams.

For moments about points such as P, Fig. 11, we proceed as follows:—

Let W_1 be the resultant load upon the portion $B_1 D_1$, Fig. 12; W_2 that upon the portion $C_1 D_1$ and W_3 that upon $C_1 A_1$.

Give the whole load system a short movement dx to the right.

Then change in bending moment at P

$$= d M_P = W_3 \tan \alpha \, dx + W_2 \tan \gamma \, dx - W_1 \tan \beta \, dx \ldots (1)$$

The bending moment at P will be a maximum when $\frac{d M_P}{dx}$ changes sign, *i.e.*, when $W_3 \tan \alpha + W_2 \tan \gamma - W_1 \tan \beta$ changes sign.

$$\text{Now } \tan \alpha = \frac{H P_1}{A_1 P_1} = \frac{\frac{ab}{l}}{a} = \frac{b}{l} \quad \ldots\ldots\ldots\ldots (2)$$

$$\tan \beta = \frac{H P_1}{B_1 P_1} = \frac{\frac{ab}{l}}{b} = \frac{a}{l} \quad \ldots (3)$$

$$\tan \gamma = \frac{E D_1 - F C_1}{C_1 D_1} = \frac{m d \tan \beta - (a - c) \tan \alpha}{d}$$

$$= \frac{m d a - (a - c) b}{d l} = \frac{a(b + c - d) - (a - c) b}{d l}$$

$$= \frac{ac - ad + bc}{dl} = \frac{c(a + b) - ad}{dl}$$

$$= \frac{cl - ad}{dl} \quad \ldots\ldots\ldots\ldots (4)$$

* *Cf.* A, p. 325.

$$\therefore \quad W_3 \tan \delta + W_2 \tan \gamma - W_1 \tan \beta$$

$$= \frac{W_3 b}{l} + \frac{W_2 (c l - a d)}{d l} - \frac{W_1 a}{l}$$

Adding and subtracting $\frac{(W_2 + W_1) b}{l}$ this becomes

$$= \frac{(W_3 + W_2 + W_1) b}{l} + W_2 \left\{ \frac{c l - a d}{d l} - \frac{b}{l} \right\} - W_1 \left(\frac{a}{l} + \frac{b}{l} \right)$$

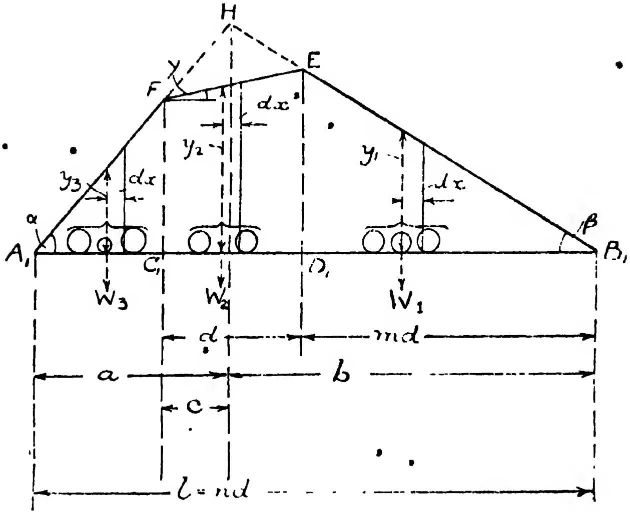


Fig. 12.—Irregular Load System on Framed Girder.

$$= \frac{W b}{l} + W_2 \left\{ \frac{c l - d (a + b)}{d l} \right\} - W_1 \left(\frac{a + b}{l} \right)$$

(where W = total load on the span)

$$= \frac{W b}{l} + W_2 \left(\frac{c - d}{d} \right) - W_1, \text{ because } (a + b) = l$$

$$= \frac{W b}{l} - \left\{ W_2 \left(\frac{d - c}{d} \right) + W_1 \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

If this expression is to become negative, $W_2 \left(\frac{d - c}{d} \right) + W_1$ must increase, and this can happen only when a load passes C or D.

As a rule $c = \frac{d}{2}$ so that expression (5) then reduces to

$$\frac{W\,b}{l} - \left\{\frac{W_2}{2} + W_1\right\}$$

The test for a maximum moment at P is therefore as follows: Place the load on the span so that the span is fully covered and with one load at D. First consider this load as part of W_1 and then as part of W_2; if this causes the expression $\frac{W\,b}{l} - \left\{W_2\left(\frac{d-c}{d}\right) + W_1\right\}$ to change sign, the position is the maximum required. If not, move the load on until another load comes at C or D.

NUMERICAL EXAMPLE.—*Take the case of a Warren Girder of* 150 *ft. span and* 10 *equal bays, and find the position of the load for a maximum moment at the centre of the fourth bay for the load system shown in Fig.* 13.

(This example is worked somewhat differently in Mr. Lea's paper, previously referred to.)

Try 10·51 tons at D, then there is 11·5 tons practically at C.

$$W = 199{\cdot}81 \text{ tons}$$

$$W_2 = 22{\cdot}01 \;\;,, \text{ taking } 10{\cdot}51 \text{ as part of } W_1$$

$$W_1 = 130{\cdot}55 \;\;,,$$

$$\frac{W\,b}{l} = \frac{199{\cdot}81 \times 97{\cdot}5}{150} = 129{\cdot}9$$

$$\frac{W_2}{2} + W_1 = 141{\cdot}55$$

$$\therefore \frac{W\,b}{l} - \left\{\frac{W_2}{2} + W_1\right\} = 129{\cdot}9 - 141{\cdot}5 = -11{\cdot}6 \text{ tons.}$$

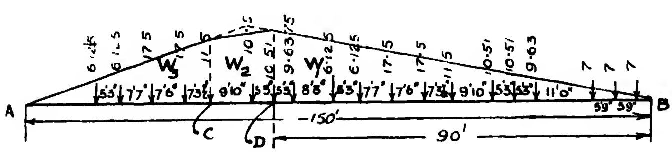


Fig. 13.

Now take the 10·51 as part of W_2.

$$W_1 = 120{\cdot}04 \text{ tons.}$$

$$W_2 = 21{\cdot}02 \;\;,,$$

$$\therefore \frac{W_2}{2} + W_1 = 130{\cdot}55$$

$$\therefore \frac{Wb}{l} - \left\{\frac{W_2}{2} + W_1\right\} = 129{\cdot}9 - 130{\cdot}5 = -{\cdot}6 \text{ tons.}$$

This does not change sign, so that the assumed position of the load does not give the maximum bending moment.

Therefore try the 9·6375 tons at D.

$$W = 199{\cdot}81 \text{ tons as before}$$

$$\therefore \frac{Wb}{l} = 129{\cdot}9$$

Treating the 9·6375 tons as part of W_1,

$$\text{we have } W_1 = 120{\cdot}04 \text{ tons}$$

$$W_2 = 21{\cdot}02 \quad ,,$$

$$\therefore \frac{W_2}{2} + W = 130{\cdot}55$$

$$\therefore \frac{Wb}{l} - \left\{\frac{W_2}{2} + W_1\right\} = -6 \text{ tons, as in the previous case.}$$

Now treat the 9·6375 tons as part of W_2; then we have

$$W_1 = 110{\cdot}40 \text{ tons approx.}$$

$$W_2 = 20{\cdot}15 \quad ,, \quad ,,$$

$$\therefore \frac{W_2}{2} + W_1 = 120{\cdot}47 \text{ tons}$$

$$\therefore \frac{Wb}{l} - \left\{\frac{W_2}{2} + W_1\right\} = 129{\cdot}9 - 120{\cdot}5 = 9{\cdot}4 \text{ tons approx.}$$

This changes sign, so that 9·6375 tons at D gives the maximum bending moment at P.

MAXIMUM SHEAR.—Let W_1 be the load on BD, W_2 that on DC and W_3 that on CA. Then shear in the panel or bay CD $= W_3 z_3 - W_2 z_2 - W_1 z_1$ (Fig 14). Let the whole load system move forward a short distance dx. The change in shear is equal to

$$dS = W_3\, dx \tan\theta - W_2\, dx \tan\phi + W_1\, dx \tan\theta \quad \ldots \quad (6)$$

$$\therefore \frac{dS}{dx} = (W_3 + W_1)\tan\theta - W_2 \tan\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$\text{Now } \tan\theta = \frac{1}{l};\ \tan\phi = \frac{K D_2 + J C_2}{C_2 D_2}$$

$$= \frac{(B_2 D_2 + A_2 C_2)\tan\theta}{C_2 D_2} = \frac{(A_2 B_2 - C_2 D_2)}{C_2 D_2 \,.\, l},$$

$$= \frac{l - d}{l d}$$

$$\frac{dS}{dx} = \frac{W_3 + W_1}{l} - W_2\left(\frac{l-d}{ld}\right)$$

$$= \frac{W_3 + W_1}{l} - \frac{W_2}{d} + \frac{W_2}{l}$$

$$= \frac{W_1 + W_2 + W_3}{l} - \frac{W_2}{d}$$

$$= \frac{W}{l} - \frac{W_2}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

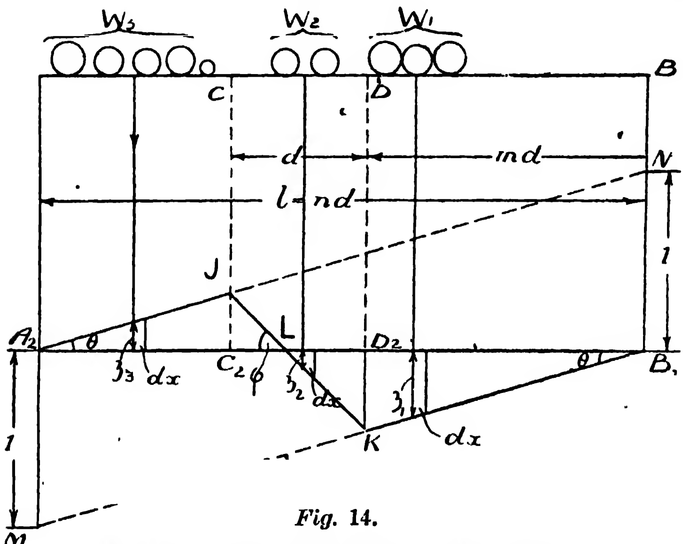


Fig. 14.

Maximum Shear for Isolated Load System.

A maximum shear occurs when $\frac{dS}{dx}$ changes sign, *i.e.*, when $\frac{W}{l} - \frac{W_2}{d}$ or $\frac{W}{n} - W_2$ changes sign.

In the limit when the loads are very close this means that $\frac{W_2}{d} = \frac{W}{l}$ or that $W_2 = \frac{W}{n}$, *i.e., the maximum shear in any bay occurs when the load on that bay is equal to the total load divided by the number of bays.** As a general rule it may be taken the

* This is equivalent to the rule proved on p. 24. The student should prove this as an exercise.

maximum shear in CD occurs when the first heavy load passes the point C.

NUMERICAL EXAMPLE.—*Take the loading shown in Fig.* 13 *on a Warren Girder of* 150 *ft. span and* 10 *equal bays and find the position of the load for a maximum shear in the* 5*th bay.*

Place the first load at the point D, Fig. 15.

$$\frac{W}{n} = \frac{119{\cdot}16}{10} = 11{\cdot}916$$

As the 6·125 passes D, $\frac{W}{n} - W_2$ does not change sign.

Now put the second load at D.

$$\frac{W}{n} = \frac{136{\cdot}66}{10} = 13{\cdot}666$$

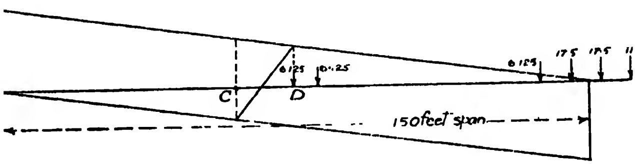


Fig. 15.

W_2 changes from 6·125 to 12·5 as the load passes D, so this does not give the maximum shear because neither of these values is greater than 13·666.

∴ Now place the first 17·5 ton load at D.

Then
$$\frac{W}{n} = \frac{148{\cdot}16}{10} = 14{\cdot}816$$

W_2 changes from 12·5 to 23·625 as the 17·5 tons passes D.

$$\therefore \frac{W}{n} - W_2 \text{ changes sign.}$$

∴ The maximum shear occurs in the bay when the first heavy load of 17·5 tons is at D.

Framed Girders with Vertical Posts.—For girders with vertical posts such as the N girder or the Pratt truss, the influence line for bending moment will be the same as for the lower flange points of the Warren girder, *i.e.*, as for an ordinary simply supported beam, and for shear in any bay will be the same as for the Warren girder, so that no further investigation of these cases appears to be necessary.

CURVED FLANGE TRUSSES.

Next consider the case of a truss with a curved flange or boom, such as is shown in Fig. 16. Considering the bars cut by a line X X, if M_C, M_F and M_G are the moments of the forces to one side of the line about the points C, F and G respectively, then, as is well known,

$$\text{Stress in CD} = \frac{M_F}{h}$$

$$\text{Stress in EF} = \frac{M_C}{n_1}$$

$$\text{Stress in FC} = \frac{M_G}{\quad}$$

We require therefore B.M. influence lines for C D, E F, and F C.

That for E F will clearly be of the same form as that for an ordinary beam (Fig. 1).

That for C D will clearly be of the same form as that for a Warren girder (Fig. 11).

Now consider the influence line for F C. Take a unit load at Q between D and B. Force on left of X X $= R_A = \frac{1 \times x}{l}$

$\therefore\ M_G = -\frac{e x}{l}$ (minus because moment is anti-clockwise).

This is a linear relation, so that influence line between B and D is a straight line B_1 J, the ordinate D_1 J being equal to $-\frac{e g}{l}$.

Next take a unit load at Q_1 between A and C, and A $Q_1 = u$.

$$R_B = \frac{u}{l}$$

$$\therefore\ M_G = R_B \times \text{BG}$$

$$= \frac{u}{l}(l + e) = u\left(1 + \frac{e}{l}\right)$$

This again is a linear relation, so that the influence line between A and C is a straight line A_1 H, the ordinate C_1 H being equal to $f\left(1 + \frac{e}{l}\right)$

As we have already proved, the influence line between C and D must also be a straight line, so that the line is completed by joining H J. It will be noted that if B_1 J and H A_1 be produced,

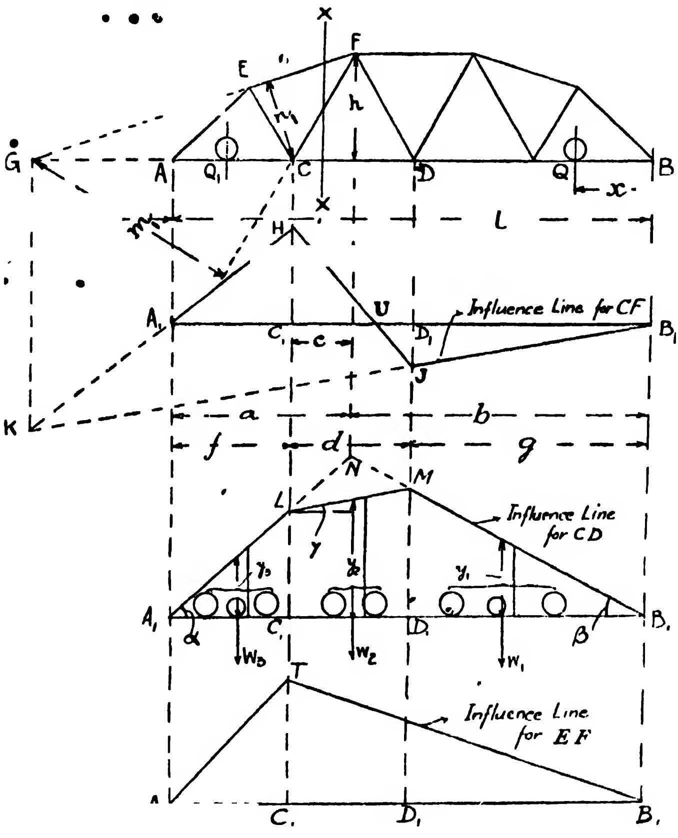


Fig 16.—Influence Lines for Curved Flange Trusses.

they will meet at K on the vertical through G because the ordinate of B_1 produced

$$= -\frac{e\,(l+e)}{r},\ i.e.,\ x = l + e$$

and that of H A_1 produced

$$= -e\left(1 + \frac{e}{l}\right) = -\frac{e(l+e)}{l}$$

This provides us with a simple method of drawing the influence line—viz., set down from A_1 a length equal to e, and join to B_1 and produce to meet vertical through G in K; join KA, and produce to meet vertical through C and join to J.

The tests of the position of the load for maximum values of M_F and M_G will be as previously determined for parallel flange trusses and simply supported beams respectively.

Maximum Stress in C F.—With a single isolated load, the maximum moment at G and therefore the maximum stress in C F clearly occurs when the load is at C.

With uniformly distributed loading, the maximum moment at G clearly occurs when the front of the load reaches the point U.

$$\text{Now } \frac{D_1 U}{C_1 D_1} = \frac{D_1 J}{D_1 J + C_1 H}$$

$$\therefore \frac{D_1 U}{d} = \frac{\frac{eg}{l}}{\frac{eg}{l} + f\left(1 + \frac{e}{l}\right)}$$

$$\therefore D_1 U = \frac{eg \cdot d}{eg + f(l + e)} \quad \text{.........................(1)}$$

Now let $l = n d$

$$g = m d$$

$$\therefore f = (n - m - 1) d$$

$$D_1 U = \frac{e m d \cdot d}{e m d + (n - m - 1) d (n d + e)}$$

(dividing through by $e d$)

$$\frac{m \cdot d}{m + (n - m - 1)\left(1 + \frac{n d}{e}\right)}$$

$$\frac{m \cdot d}{m + n - m - 1 + \frac{(n - m - 1) n d}{e}}$$

$$= \frac{m \cdot d}{\left\{(n - 1) + \frac{(n - m - 1)}{e} \cdot n d\right\}} \quad \text{.............(2)}$$

Comparing this with the result obtained on page 23, we see that the result is similar and contains the additional term $\frac{(n - m - 1)nd}{e}$ in the denominator. If the flanges are parallel, then $e =$ infinity and the additional term reduces to zero.

For an isolated load system, a similar treatment to that for the previous cases shows that the maximum moment occurs when $\frac{W}{l} - \frac{W_2}{d}\left(1 + \frac{f}{e}\right)$ changes sign.

CHAPTER III.

INFLUENCE LINES FOR FIXED AND CONTINUOUS BEAMS.

FIXED OR BUILT-IN BEAMS.

SUPPOSE that an isolated load W is placed at a point P on a beam A B, Fig. 17, whose ends are fixed horizontally, the beam being of constant cross section. Then if G is the centroid of the Δ A H B, which is the free bending moment diagram, and the end bending moment diagram A D E B be divided into two Δs whose centroids lie upon the 'third lines' X X and Y Y, we have to satisfy the following conditions: *

(1) The area A D E B = area A H B.

(2) The centroid of A D E B lies on the vertical through G, $\therefore$ taking moments round Y Y we get

Area Δ A H B $\times$ z = moment of Δ A D B about Y Y

$$= \text{area } \Delta \text{ ADB} \times \frac{l}{3}$$

$$\text{Now } z = \frac{l}{6} - \text{CT}$$

$$= \frac{l}{6} - \frac{1}{3}\text{CP}$$

$$\frac{l}{6} \quad \frac{1}{3}\left(\frac{l}{2} - b\right)$$

$$\frac{b}{3}$$

$$\text{Area } \Delta \text{ AHB} = \frac{\text{AB} \cdot \text{PH}}{2} = \frac{\frac{1}{2} l . \frac{\text{W}ab}{l}}{l} = \frac{\text{W}ab}{2}$$

$$\text{Area } \Delta \text{ ADB} = \frac{\text{M}_\text{A} . l}{2}$$

$$\therefore \text{M}_\text{A} \frac{l}{2} \cdot \frac{l}{3} = \frac{\text{W}ab}{2} \cdot \frac{b}{3}$$

* For proof see A, p. 231.

$$i.e. \quad M_A = \frac{W a b^2}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now $\frac{(M_A + M_B)\, l}{2}$ = area Δ A H B $= \frac{W a b}{2}$

$$\therefore \frac{M_B\, l}{2} = \frac{W a b}{2} - \frac{M_A\, l}{2}$$

$$= \frac{W a b}{2} - \frac{W a b^2}{2 l}$$

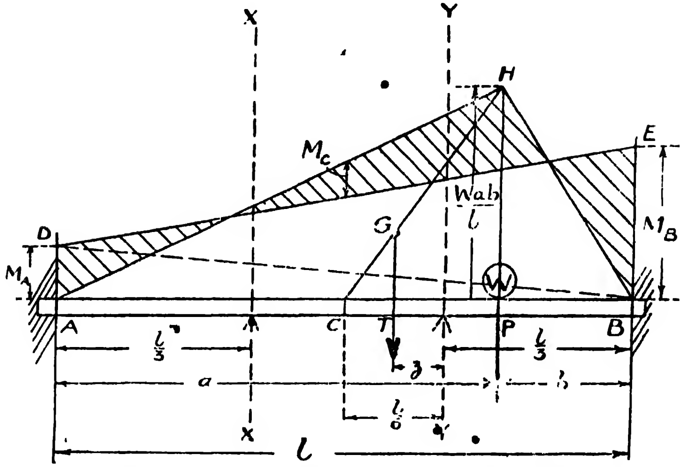


Fig. 17.—*Fixed Beam with Isolated Load.*

$$= \frac{W a b}{2}\left(1 - \frac{b}{l}\right) = \frac{W a b}{2}\left(\frac{l - b}{l}\right)$$

$$= \frac{W a b}{2} \cdot \frac{a}{l}$$

$$\therefore \; M_B = \frac{W a^2 b}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

(This result could have been written direct from (1) from conditions of symmetry.)

Now calculate the bending moment M_C at the centre.

$$M_C = \frac{W b}{l} \cdot \frac{l}{2} - \frac{M_A + M_B}{2}$$

$$= \frac{W b}{2} - \frac{W a b}{2 l}$$
$$= \frac{W b}{2}\left(1 - \frac{a}{l}\right)$$
$$= \frac{W b}{2 l}\left(\frac{l - a}{l}\right)$$
$$= \frac{W b^2}{2 l} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

This holds up to the value $b = \frac{l}{2}$; beyond this we must work from the other end or use the above equation and take b the distance to the nearer end. We get the following results for M_A and M_B from the above equations for a unit load.

a	b	M	M_B	$-\left(\frac{M_B - M_A}{l}\right)$
$\cdot 1\, l$	$\cdot 9\, l$	$\cdot 081\, l$	$\cdot 009\, l$	$\cdot 072$
$\cdot 2\, l$	$\cdot 8\, l$	$\cdot 128\, l$	$\cdot 032\, l$	$\cdot 096$
$\cdot 3\, l$	$\cdot 7\, l$	$\cdot 147\, l$	$\cdot 063\, l$	$\cdot 084$
$\cdot 4\, l$	$\cdot 6\, l$	$\cdot 144\, l$	$\cdot 096\, l$	$\cdot 048$
$\cdot 5\, l$	$\cdot 5\, l$	$\cdot 125\, l$	$\cdot 125\, l$	$\cdot 000$
b	a	M_B.	M_A	$+\left(\frac{M_B - M_A}{l}\right)$

The maximum value of M_B is given by putting $\frac{d M_B}{d a}$

$$i.e. \quad \frac{d\{a^2 (l - a)\}}{d a} = 0$$
$$i.e. \quad 2 a l - 3 a^2$$
$$\text{or} \qquad a = \frac{2 l}{3} \qquad (4)$$

Then, $M_B \quad \frac{4 l}{27}$ for unit load.

The results of the above table are plotted in Fig. 18; the ordinates of this figure at any point gives the end bending moments when a unit load is placed at that point, *the curves*

$A_1 d B_1$ *and* $A_1 e B_1$ *are therefore influence lines for the end bending moments* and can be drawn once and for all for any given span.

If therefore we have a number of loads W_1, W_2, &c., upon a fixed beam we first draw the free bending moment diagram in the ordinary simple manner and read off the ordinates $c\,d$, $c\,e$ at each load, then

$$M_A = \Sigma W \,.\, cd$$
$$M_B = \Sigma W \quad ce$$

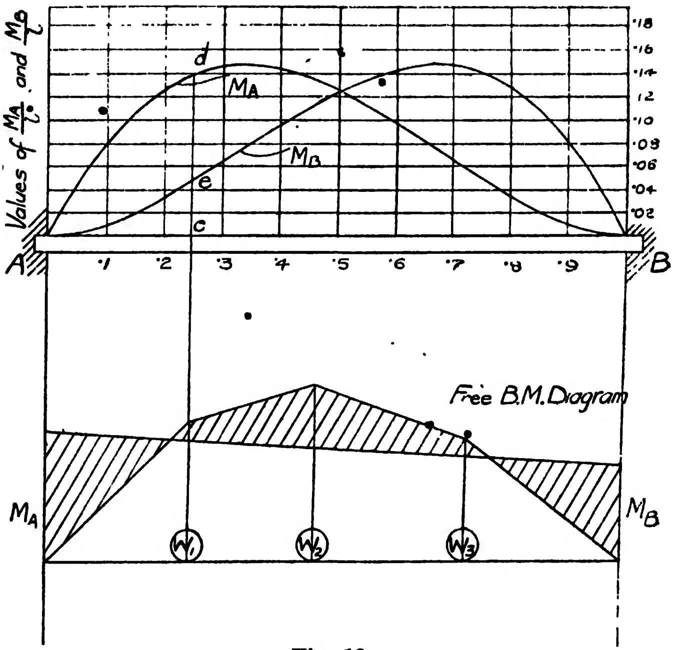


Fig. 18.

Influence Lines for End B.Ms. of Fixed Beams.

These end bending moments are then set up to the scale to which the free bending moment diagram is drawn and the resulting bending moment diagram is obtained as shown shaded in the figure.

INFLUENCE LINE FOR CENTRE BENDING MOMENT.—We have seen that by equation (3)

$$M_C = \frac{W\,b^2}{2\,l}$$

For unit load therefore $M_C = \dfrac{b^2}{2\,l}$.

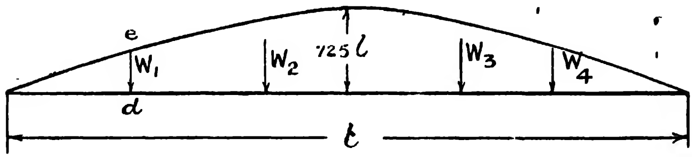


Fig. 19.
Influence Line for Centre B.M. of Fixed Beams.

We see from this that the influence line for the bending moment at the centre of a fixed beam is a parabola as shown in Fig. 19, whose height at the centre is ·125 l; it will be found convenient to draw this parabola to an enlarged vertical scale.

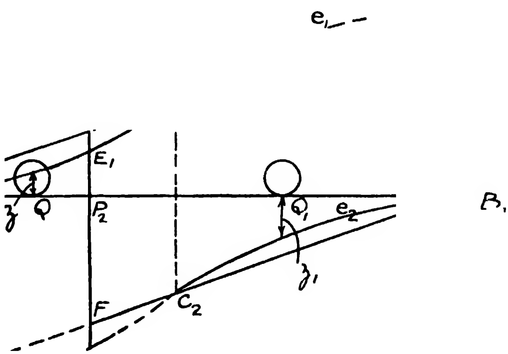


Fig. 20.
Shear Influence Line for Fixed Beams.

For a number of loads W_1 W_2 W_3 W_4, &c., therefore the bending moment at the centre $= \Sigma\, W_1 \,.\, d\,e$.

SHEAR INFLUENCE LINE.—The shear at any point between A and P (Fig. 17) is given by $S = \frac{W b}{l} - \frac{M_B - M_A}{l}$.

We can then by aid of the last column of the table on p. 36 construct a shear influence line as follows for the beam with fixed ends.

First draw the shear influence line $A_2\,E\,F\,B_2$ (Fig. 20), for the simply supported beam; then draw the curves $B_2\,e_2\,C_2\,F_1\,K$ and $A_2\,E_1\,C_1\,e_1\,J$ whose ordinates are equal to $\frac{M_B - M_A}{l}$ measured from the inclined lines $B_2\,K$ and $A_2\,J$. The shear influence line for the point P_2 on the fixed beam is then $A_2\,E_1\,F_1\,e_2\,B_2$.

If therefore a load W is placed at the point Q, the shear at P_2 is $+\ W z$; if it is placed at Q_1, then the shear is $-\ W z_1$.

CONTINUOUS BEAMS.

Two Equal Spans.—We will consider in the first place the case of a continuous beam A B C (Fig. 21) of two equal spans and of constant cross sections, the supports being all at the same level.

If a single isolated load W is placed at a point Q on the first span at distance $a = \alpha\, l$ from the point A, we have by the *Theorem of Three Moments*,* since there is no load on B C

$$M_A \,.\, l + 2\, M_B\,(l + l) + M_C\, l = \frac{6\, S y}{l}$$

$$= \frac{6 \times \text{moment of } \Delta\, A_1\, D\, B_1 \text{ about } A_1}{l}$$

$\therefore$ since M_A and $M_C = 0$

$$4\, M_B \,.\, l = \frac{6\, l}{l} \,.\, \frac{W\, \alpha\,(1 - \alpha)\, l}{2} \left\{ \alpha\, l + \frac{2}{3}\left(\frac{l}{2} - \alpha\, l\right)\right\}$$

$$= \frac{3\, l\, W\, \alpha\,(1 - \alpha)\,(1 + \alpha)\, l}{3}$$

$$\therefore\ M_B = \frac{W\, \alpha\,(1 - \alpha^2)\, l}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

* For general proof *see* A, p 251.

$$\therefore \quad R_A = W(1 - a) - \frac{(M_B' - M_A)}{l}$$

$$= W(1 - a) - \frac{W a (1 - a^2)}{4}$$

$$= W(1 - a)\left\{1 - \frac{a(1 + a)}{4}\right\}$$

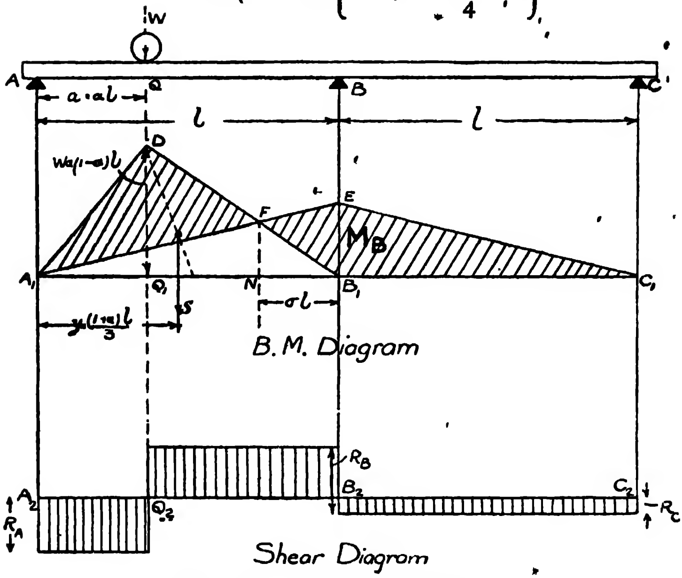


Fig. 21.—*Continuous Beams.*

$$= \frac{W(1 - a)}{4}\{4 - a(1 + a)\} \qquad .(2)$$

$$R_C = 0 - \frac{(M_B - M_C)}{l}$$

$$= -\frac{W a (1 - a^2)}{4} \qquad .(3)$$

$$R_B = W a + \frac{(M_B - M_C)}{l} + \frac{(M_B - M_A)}{l}$$

$$= W a + \frac{W a (1 - a^2)}{2}$$

$$= \frac{W a}{2}(3 - a^2) \;\ldots\ldots\ldots\ldots\ldots\ldots \qquad .(4)$$

These equations enable us to draw the bending moment and shear diagrams shown in Fig. 21; it will be noted that the reaction at the end C is negative, thus showing that this end must be held down to keep it upon its support.

The point F of contraflexure is important; suppose that it occurs at distance σl from the support B.

Then F N (from consideration of free B.M. diagram)

$$= W a . \sigma l \quad \dots\dots\dots\dots \quad (4a)$$

$$\text{also } \frac{F N}{E B_1} = \frac{A_1 N}{A_1 B_1}$$

$$\therefore F N = \frac{M_B . (1 - \sigma) l}{l} = \frac{W a (1 - a^2) l (1 - \sigma)}{4}$$

$$\therefore \sigma l = \frac{(1 - a^2) l (1 - \sigma)}{4} \text{ (from (4a))}$$

$$4 \sigma = (1 - a^2)(1 - \sigma)$$

$$\therefore 4 \sigma + \sigma (1 - a^2) = (1 - a^2)$$

$$\therefore \sigma = \frac{1 - a^2}{5 - a^2} \quad \dots\dots\dots\dots \quad (5)$$

The point of inflection is always between the load and the centre support. As a varies, σ varies and has a maximum value determined by $a = 0$, because a^2 will always be positive.

$$\therefore \sigma \text{ max.} = \frac{1}{5} \quad \dots\dots\dots\dots \quad (6)$$

INFLUENCE LINE FOR M_B.—The results of equation (1) for M_B for unit loads may be tabulated as follows:—

a	M_B	a	M_B
·1	·025 l	·6	·096 l
·2	·048 l	·7	·089 l
·3	·068 l	·8	·072 l
·4	·084 l	·9	·043 l
·5	·094 l		

The maximum value of M_B is given by

$$\frac{d\,a\,(1 - a^2)}{d\,a} = 0$$

i.e. $1 - 3\,a^2 = 0$

or $a = \frac{1}{\sqrt{3}} = \cdot577$

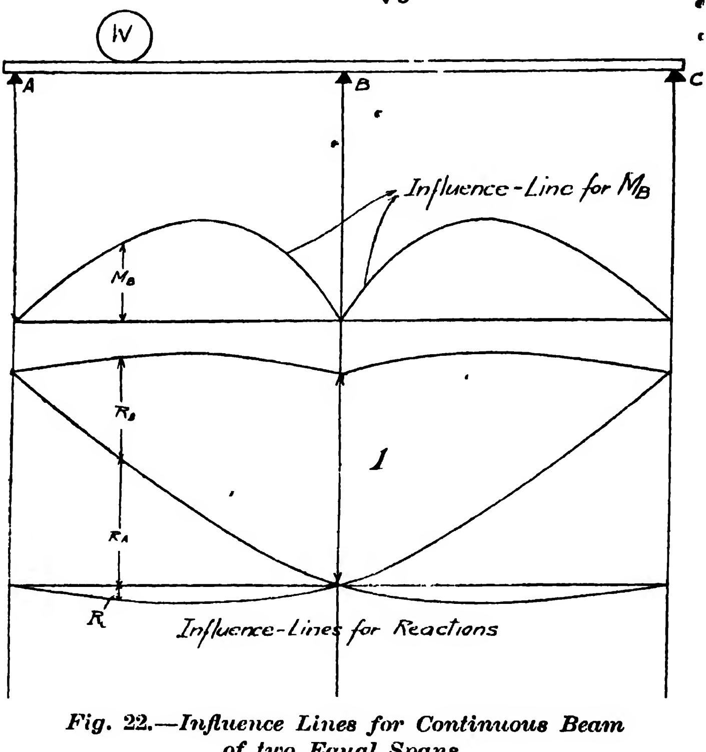


Fig. 22.—Influence Lines for Continuous Beam of two Equal Spans.

Then for unit load

$$M_B = \cdot577\,(1 - \tfrac{1}{3})\,l$$

$$= \frac{\cdot577\,l}{6} = \cdot0962\,l = \frac{l}{10.39}$$

These results can then be plotted as shown in Fig. 22, and give the influence line for the moment at the central support.

INFLUENCE LINE FOR R_A and R_C.—From equations (2) and (4) we obtain the following values of R_A and R_C for various values of a for unit load.

a	R_A	R_C	a	R_A	R_C
·1	·875	− ·025	·6	·304	− ·096
·2	·752	− ·048	·7	·211	− ·088
·3	·632	− ·068	·8	·128	− ·072
·4	·516	− ·084	·9	·057	− ·043
·5	·406	− ·094			

The influence line for these reactions therefore comes as shown in Fig. 22.

UNIFORMLY DISTRIBUTED LOAD.—Take first the case of one span fully covered. It follows from considerations of symmetry in this case that M_B = one-half that when both spans are loaded, *i.e.* $M_B = \frac{p\,l^2}{16}$, or from the Theorem of Three Moments

$$0 \times l + 2\,M_B\,(l_2 + l) + 0 \times l = 6\left\{\frac{2}{3}\cdot\frac{l\,p\,l^2}{8}\cdot\frac{l}{2} \div l + 0\right\}$$

$$i.e.\ 4\,M_B\,.\,l = \frac{p\,l^3}{4}$$

$$\text{or } M_B = \frac{p\,l^2}{16}$$

$$\text{Then } R_A = \frac{p\,l}{2} - \frac{p\,l^2}{16\,l} = \frac{7\,p\,l}{16}$$

$$R_C = 0 - \frac{p\,l^2}{16\,l} = -\frac{p\,l}{16}$$

The bending moments and shear diagrams then come as shown in Fig. 23.

We have shown that the maximum value of the point of contraflexure is $\frac{1}{5}l$ from the centre support, for points therefore less than $\frac{4l}{5}$ from either end, there is no point of contraflexure for any position of the load, so that the maximum positive bending moment for such points occurs when the span containing the point is fully loaded, and the maximum negative bending

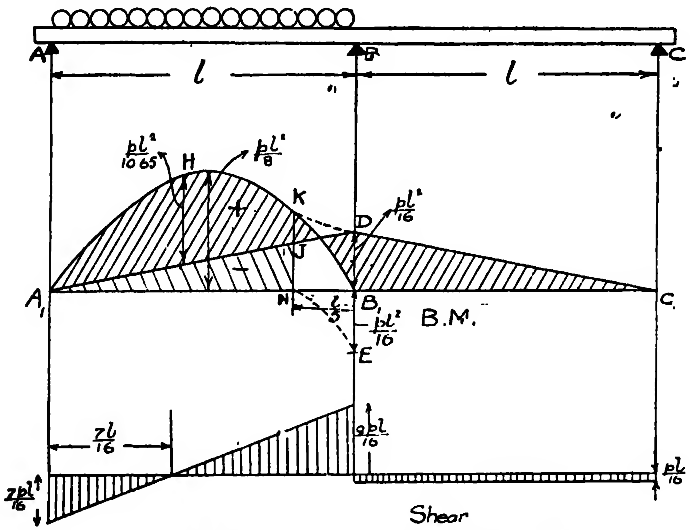


Fig. 23.—Continuous Beam of two Equal Spans— Uniform Load.

moment occurs when the other span is loaded. For points therefore from A_1 to N, Fig 23, the curves A_1 H K and the straight line A_1 N give the maximum positive and negative bending moments, the ordinates being measured from the line A_1 J.

For points between N and B_1, the maximum positive bending moments occur when N B_1 only is loaded, and the maximum negative bending moments occur when A_1 N and B_1 C_1 are loaded. The dotted lines K D and N E give the maximum positive and

negative moments for these points, ordinates being measured as before from the line J D.

***Influence Lines for two Unequal Spans—Dr. Lea's Treatment.**—In Vol. CLXXXV. of *Proc Inst. C.E.*, Dr. F. C. Lea in a very valuable paper gives the following treatment of influence lines for continuous beams :—

Take the beam shown in Fig. 24.

At the point P

$$M = EI\frac{d^2y}{dx^2} = R_A x - W_1(x - a_1) + R_B(x - l_1) - W_2(x - c) \ldots (1)$$

Integrating twice.

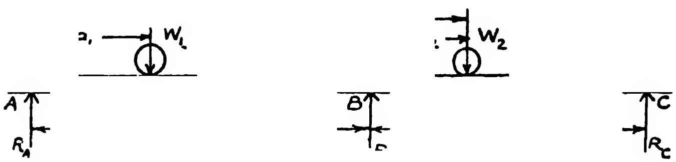


Fig. 24.

$$y = \frac{R_A x^3}{6} - \frac{W_1(x - a_1)^3}{6} + \frac{R_B(x - l_1)^3}{6} - \frac{W_2(x - c)^3}{6} + Cx + D \ldots (2)$$

$$y = 0 \text{ for } x = 0$$

$$\therefore \quad D = 0$$

$$y = 0 \text{ for } x = l_1$$

$$0 = \frac{R_A l_1^3}{6} - \frac{W_1(l_1 - a_1)^3}{6} + C l_1$$

$$\frac{W_1(l_1 - a_1)^3}{6 l_1} - \frac{R_A l_1^2}{6} \qquad \cdot (3)$$

$$y = 0 \text{ for } x = (l_1 + l_2)$$

$\therefore$ (multiplying through by 6)

$$R_A(l_1 + l_2)^3 - W_1(l_1 + l_2 - a_1)^3 + R_B l_2^3$$

$$-W_2(l_1 + l_2 - c)^3 + \frac{W_1}{6 l_1}(l_1 - a_1)^3(l_1 + l_2) - \frac{R_A l_1^2(l_1 + l_2)}{6} = 0 \ldots (4)$$

Taking moments round B we have

$$R_C\, l_2 - W_2\,(c - l_1) = R_A\, l_1 - W_1\,(l_1 - a_1) \quad \text{.........(5)}$$

Also $\quad R_A + R_B + R_C = W_1 + W_2 \quad$ ....(6)

Then solving (4), (5), and (6), and putting $c = l_1 + a_2$, we get after reduction

$$R_A = \frac{W_1}{l_1\,(l_1 + l_2)}\left\{ l_1^2 + l_1\, l_2 - \frac{3}{2}\, a_1\, l_1 - a_1\, l_2 + \frac{a_1^3}{2\, l_1}\right\}$$

$$- \frac{W_2}{l_1\,(l_1 + l_2)}\left\{ a_2\, l_2 - \frac{3}{2}\, a_2^2 + \frac{a_2^3}{2\, l_2}\right\} \quad \text{..................(7)}$$

Alternative Treatment for R_A.—We could get the same result as Dr. Lea deduces as above by the following treatment, which follows more upon the lines of our previous working.

Suppose that in Fig. 21 the spans are unequal and equal to l_1 and l_2 respectively.

Then applying the Theorem of Three Moments we shall get in a similar manner to that dealt with on p. 39 for a load W_1 on the first span,

$$2\, M_B\,(l_1 + l_2) = W_1\, a\,(1 - a^2)\, l_1^2$$

$$\therefore M_B = \frac{W_1\, a\,(1 - a^2)\, l_1^2}{2\,(l_1 + l_2)}$$

$$\therefore R_A = W_1\,(1 - a) - \frac{M_B - M_A}{l}$$

$$= W_1\,(1 - a) - \frac{W_1\, a\,(1 - a^2)\, l_1}{2\,(l_1 + l_2)}, \text{ since } M_A = 0$$

$$= \frac{W_1}{(l_1 + l_2)}\left\{(1 - a)\,(l_1 + l_2) - \frac{(a - a^3)\, l_1}{2}\right\}$$

$$= \frac{W_1}{(l_1 + l_2)}\left\{l_1 + l_2 - a\, l_1 - a\, l_2 - \frac{a\, l_1}{2} + \frac{a^3\, l_1}{2}\right\}$$

$$= \frac{W_1}{(l_1 + l_2)}\left\{l_1 + l_2 - \frac{3\, a\, l_1}{2} - a\, l_2 + \frac{a^3\, l_1}{2}\right\}.$$

Then putting $a = \dfrac{a_1}{l_1}$ we get

$$R_A = \frac{W_1}{l_1\,(l_1 + l_2)}\left\{l_1^2 + l_1 l_2 - \frac{3}{2}\, a_1 l_1 - a_1\, l_2 + \frac{a_1^3}{2\, l_1}\right\}$$

To deal with the load on the second span, we may find R_C for a load W_2 on the 1st span, reverse l_1 and l_2 and put a_2 for $(1 - a)$, then the R_C thus obtained will be equal to the R_A required.

$$\text{Then } R_C = \frac{M_B - M_C}{l_1} = \frac{-W_2(1 - a_2) \cdot a_2 (2 - a_2) l_2^2}{2(l_1 + l_2) l_1}$$

$$= \frac{-W_2 a_2 l_2^2}{l_1(l_1 + l_2)} \left\{ \frac{2 - 3a_2 + a_2^2}{2} \right\}$$

$$= \frac{W_2}{l_1(l_1 + l_2)} \left\{ a_2 l_2 - \frac{3a_2^2}{2} + \frac{a_2^3}{2 l_2} \right\}, \text{ putting } a_2 = \frac{a_2}{l_2}$$

We get therefore as our combined result as before

$$R_A = \frac{W_1}{l_1(l_1 + l_2)} \left\{ l_1^2 + l_1 l_2 - \frac{3}{2} a_1 l_1 - a_1 l_2 + \frac{a_1^3}{2 l_1} \right\}$$

$$- \frac{W_2}{l_1(l_1 + l_2)} \left\{ a_2 l_2 - \frac{3a_2^2}{2} + \frac{a_2^3}{2 l_2} \right\} \quad \ldots\ldots\ldots\ldots(7)$$

Graphical Construction for R_1.—At the central support make BD $= l_1 l_2$, Fig. 25. Join CD and produce it to E to meet the support vertical at A. Then the equation to DE in terms of the length a_1 as a variable is given by

$$y = ac = l_1(l_1 + l_2 - a_1) = l_1^2 + l_1 l_2 - a_1 l_1$$

Also in terms of a_2 as a variable the portion CD is given by

$$y = ef = l_1 l_2 - l_1 a_2, \text{ because its slope } = l_1$$

Now draw a curve A b D given by

$$y = ab = \frac{1}{2} a_1 l_1 + a_1 l_2 - \frac{a_1^3}{2 l_1} \quad \ldots\ldots\ldots\ldots(8)$$

$$\text{Then } bc = ac - ab = l_1^2 + l_1 l_2 - \frac{3 a_1 l_1}{2} - a_1 l_2 - \frac{a_1^3}{2 l_1}$$

$$\therefore \text{ Since AE } = l_1(l_1 + l_2)$$

$$R = \frac{W_1 \cdot bc}{\text{AE}} \text{ for load } W_1$$

Next draw a curve D g C given by

$$y = eg = a_2 l_2 - \frac{3a_2^2}{2} + \frac{a_2^3}{2 l_2} + l_1 l_2 - l_1 a_2 \quad \ldots\ldots\ldots(9)$$

Then $fg = eg - ef = l_2 a_2 - \frac{3 a_2^2}{2} + \frac{a_2^3}{2 l_2}$

$$\therefore \ R_A = \frac{W_1 . fg}{AE} \text{ for load } W_1$$

$\therefore$ for a number of loads on each span

$$R_A = \frac{1}{AE} \Sigma \left\{ W_1 . bc - W_2 . fg \right\} \quad \text{............(10)}$$

Or treating bc as $+ve$ and fg as $-ve$

$$AE . R_A = \Sigma W \text{ (ordinate between curve and line CE)}$$

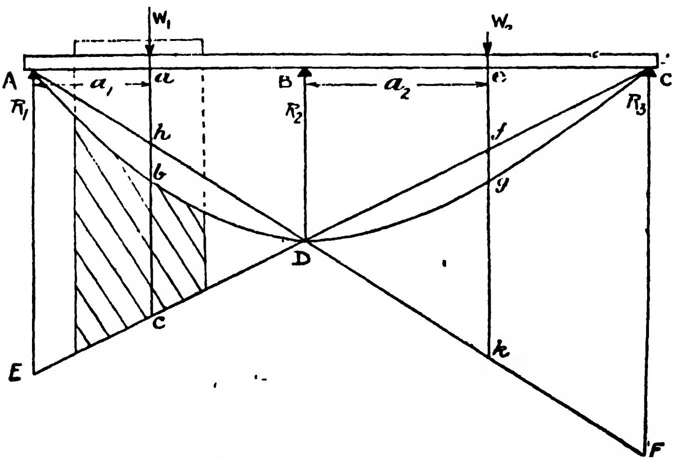


Fig. 25.—Influence Lines for Continuous Beams of two Unequal Spans.

VALUE OF R_C.—If we join A D and produce to meet the support vertical at C in F we shall get by exactly similar reasoning:

$$R_C = \frac{W_2 . gk - W_1 . bh}{CF}$$

i.e. $CF . R_C = \Sigma W$ (ordinate between curve and line A F).

The diagram in Fig. 25 gives us therefore an influence line for the reactions.

UNIFORMLY DISTRIBUTED LOAD.—From the general nature of influence lines it follows that for a uniform load of intensity p

$$R_A = \frac{p}{AE} \times \text{area under load.}$$

It is clear from the figure that for a uniform load of given length, the positive reaction at A is a maximum when the load starts at A and that the negative reaction at A is a maximum when the area of the piece under D C is a maximum; in most cases this position can be found with sufficient accuracy by trial.

When the length of the load is greater than either span, the maximum positive reaction at A occurs when the span A B is covered and the maximum negative reaction occurs when B C is covered.

Scales.—Suppose the space scale is 1″ = x feet and A E = y actual inches.

Then $R_A = \frac{p\,x}{y}$ (area in square inches).

***Tables to facilitate Drawing of Curves.**—We have prepared the following tables to facilitate the drawing of the curves A b D and D g C, which we will refer to as the 'First span curve' and 'Second span curve' respectively. In these tables the first span is always taken as the greater; when such is not the case we have only to work from the other end, *i.e.*, look upon the girder from the opposite side.

It is interesting to compare the curves for equal spans with the figures given on p. 46.

Take, for instance, $l_1 = l_2 = l$ and $a_1 = \frac{l}{2}$

The table gives $y = {\cdot}688\,l^2 = a\,b$

Now $a\,c$ for $a_1 = \frac{l}{2} = 1{\cdot}5\,l^2$ because B D $= l^2$

$$\therefore\ R_A = \frac{W\,.\,b\,c}{AE} = \frac{W\,(1{\cdot}5 - {\cdot}688)\,l^2}{2\,l^2} = {\cdot}406\ W$$

This is the value given on p. 43.

TABLE FOR FIRST SPAN CURVE (Equation 8).

Values of a	Values of $\frac{y}{l_1^2}$ $l_2 = l_1$	$l_2 = ·9\,l_1$	$l_2 = ·8\,l_1$	$l_2 = ·7\,l_1$	$l_2 = ·6\,l_1$	$l_2 = ·5\,l_1$	$l_2 = ·4\,l_1$	$l_2 = ·3\,l_1$		
$·1\,l_1$	·150	·140	·130	·120	·110	·100	·090	·080	·076	·060
$·2\,l_1$	·296	·276	·256	·236	·216	·196	·176	·156	·136	·116
$·3\,l_1$	·437	·407	·377	·347	·317	·287	·257	·227	·197	·167
$·4\,l_1$	·568	·528	·488	·448	·408	·368	·328	·288		·208
$·5\,l_1$	·688	·638	·588	·538	·488	·438	·388	·338		·238
$·6\,l_1$	·792	·732	·672	·612	·552	·492	·432	·372		·252
$·7\,l_1$	·879	·809	·739	·669	·599	·529	·459	·389	·339	·249
$·8\,l_1$	·944	·864	·784	·704	·624	·544	·464	·384	·304	·224
$·9\,l_1$	·985	·895	·805	·715	·625	·525	·445	·355	·265	·175
	1·000	·900	·800	·700	·600	·500	·400	·300	·200	·100

TABLE FOR SECOND SPAN CURVE (Equation 9).

Values of a_2	Values of $\frac{y}{l_1^2}$ $l_2 = l_1$	$l_2 = ·9\,l_1$	$l_2 = ·8\,l_1$	$l_2 = ·7\,l_1$	$l_2 = ·6\,l_1$	$l_2 = ·5\,l_1$	$= ·4\,l_1$	$= ·3\,l_1$	$l_2 = ·2\,l_1$	$l_2 = ·1\,l_1$
·0	1·000	·900	·800	·700	·600	·500	·400	·300	·200	·100
$·1\,l_2$	·985	·879	·774	·673	·571	·471	·374	·278	·183	·091
$·2\,l_2$	·944	·836	·732	·630	·531	·435	·342	·253	·166	·081
$·3\,l_2$	·879	·775	·674	·577	·484	·394	·308	·225	·146	·071
$·4\,l_2$	·792	·694	·602	·514	·435	·348	·271	·196	·127	·061
$·5\,l_2$	·688	·601	·520	·441	·367	·297	·227	·168	·107	·051
$·6\,l_2$	·568	·496	·427	·363	·301	·242	·185	·135	·086	·041
$·7\,l_2$	·437	·380	·328	·277	·325	·184	·141	·102	·067	·030
$·8\,l_2$	·396	·258	·228	·191	·156	·126	·097	·069	·046	·020
$·9\,l_2$	·150	·130	·115	·097	·077	·064	·049	·035	·025	·010

Shear Influence Line.—Take any point P, Fig. 26, of a continuous beam of two spans, not necessarily equal.

$$\text{Shear at P} = S_P = R_A - \sum_A^P W$$

$$= \sum_A^C \frac{W \cdot bc}{AE} - \sum_A^P W . \qquad (12)$$

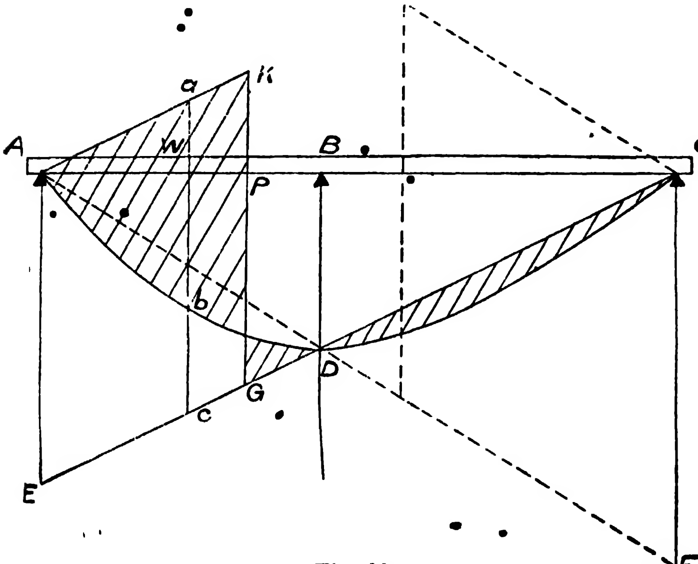


Fig. 26.
Shear Influence Line for Continuous Beam of two Spans.

If we treat AE as unity we may write this as

$$S_P = \sum_A^P W\,bc + \sum^C W\,bc - \sum^P W$$

$$= \sum_A^P W\,(bc - 1) + \sum_P^C W \cdot bc \ldots\ldots \qquad (13)$$

Draw AK parallel to EC; then ac = AE = unity.

$$\therefore S_P = \sum^P W \cdot ab + \sum_P^C W \cdot bc$$

∴ A K G C *is the shear influence line, the ordinates being measured from the curve.*

A similar construction, indicated by dotted lines, gives the shear influence line for the span B C, C F being used in place of A E.

MOVING LOADS.—Since the area under the influence line gives the shear for a uniformly distributed load, it will be clear from Fig. 26 and the signs of the areas shown thereon, that the maximum negative shear at P for such a load occurs when the portions A P and B C are fully covered with the load and the maximum positive shear when the portion P B only is covered, this condition being of course impossible with a load of greater length than P B.

The scales to which these areas are to be read will be as explained for the bending moment influence line.

With an isolated load system the position of the load to give maximum shear at any point can be found by trial with the load system set out on tracing paper. The maximum of the various maxima will usually be found for positive shear to come at A and to occur when the heaviest load is at A; the maximum for negative shear usually comes at B and occurs when the heaviest load is at B.

* **Bending Moment Influence Line.**—Lett he point P, Fig. 27, be at distance z from the left-hand support A.

Then bending moment at P, W being applied at a_1 from A

$$= M_1 = R_A z - \sum_A^P W(z - a_1)$$

$$= z\left\{\sum_A^C \frac{W.ac}{AE} - \sum_A^P \frac{W(z-a_1)}{z}\right\} \quad \ldots\ldots (14)$$

Now draw through P a vertical P G and join A G.

Then $$\frac{bc}{AE} = \frac{z - a_1}{z}$$

$\therefore$ Regarding A E as unity we have

$$M_P = z\left\{\sum_A^C W.ac - \sum_A^P W.bc\right\}$$

$$= z\left\{\sum_A^P W.ac + \sum_P^C W.ac - \sum_A^P W.bc\right\}$$

$$= z\left\{\sum_A^P W(ac - bc) + \sum_P^C W.ac\right\}$$

$$= z \left\{ \sum_{A}^{P} W \cdot ab + \sum_{P}^{C} W \cdot ac \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

$$= z \sum (W \times \text{ordinate between } AG, GC \text{ and the curve}).$$

∴ A G C *is a bending moment influence line, the ordinates being measured from the curve.*

In this case, as before, the ordinates are considered positive when below the curve and negative when above.

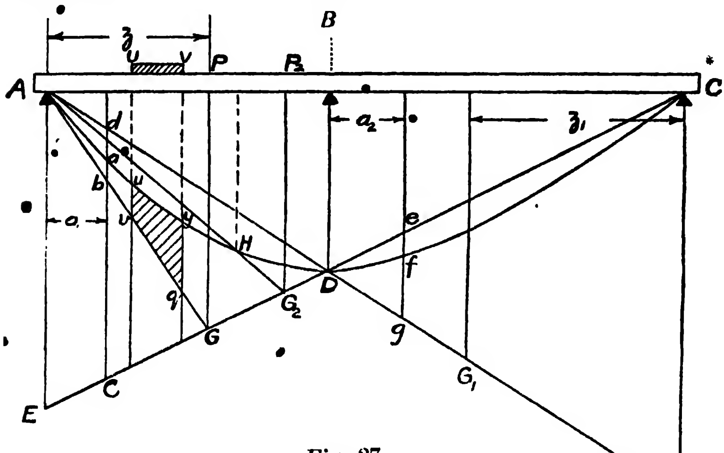


Fig. 27.

B.M. Influence Line for Continuous Beam of two Spans.

For a point on the span B C the influence line will be $z_1 \Sigma$ ($W_1 \times$ ordinate between A G_1, G_1 C and the curve), the length C F being treated as unity in this case.

Uniformly distributed Load.—Let a uniformly distributed load of intensity p cover the portion U V of the span.

Then, it follows from the properties of influence lines previously considered that due to this load we have:

$$M_P = p\,z \times \text{area } uvqy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

Scales.—As we have treated A E as unity and A E $= l_1 (l_1 + l_2)$ the value of M_P may be written

$$M_P = \frac{p\,z}{l_1 (l_1 + l_2)} \times \text{area } uvqy$$

this area for use in equation (15a) must then be read to a scale

$$1 \text{ sq. in} = \frac{x}{y}$$

where y = A E in actual inches

x = space scale of feet per inch.

Suppose for instance that $l_1 = 25$ feet and $l_2 = 20$ feet and the space scale is $1'' = 4$ feet; then $l_1 l_2 = 500$ and suppose this is set down to the scale $1'' = 200$ sq. ft. (*i.e.* B D = $2\frac{1}{2}''$). If the area comes A actual inches, then

$$M_P = \frac{p z x A}{y} = \frac{4 p z A}{5 \cdot 625}$$

Maximum Bending Moments.—It will be clear from Fig. 27 that the maximum positive bending moment at P occurs with a uniformly distributed load when the span A B only is covered and the maximum negative bending moment occurs when B C only is covered by the load. For points such as P_2 where A G_2 cuts the curve, the ordinates are positive only for points between H and D; in this case with a uniformly distributed load the maximum positive bending moment occurs when the load extends from B to the point above H, and the maximum negative bending moment occurs when the remainder of the two spans is covered.

We will now find the limits within which this reversal occurs; the limits of the point P_2 are clearly B on one side, and the intersection of the span of the vertical through the point where the tangent to the curve A a D cuts C E on the other; we will assume that P is this point.

Now the curve A a D is given by (eq. 8, p. 47)

$$y = \frac{1}{2} a_1 l_1 + a_1 l_2 - \frac{a_1^3}{2 l_1}$$

$$\therefore \frac{dy}{d a_1} = \frac{1}{2} l_1 + l_2 - \frac{a_1^2}{l_1}$$

When $a_1 = 0$, $\frac{dy}{d a_1}$ = slope of curve at A.

$$= \frac{1}{2} l_1 + l_2$$

$$\therefore \text{PG} = z\left(\frac{1}{2}l_1 + l_2\right)$$

$$\text{Also } \frac{\text{PG}}{\text{PC}} = \frac{\text{BD}}{\text{BC}} = \frac{l_1 l_2}{l_2} = l_1$$

$$\therefore \text{PG} = l_1 \text{PC} = l_1(l_1 + l_2 - z)$$

$$\therefore z\left(\frac{1}{2}l_1 + l_2\right) = l_1(l_1 + l_2) - l_1 z$$

$$\text{or } z = \frac{l_1(l_1 + l_2)}{\frac{3l_1}{2} + l_2} \qquad (16)$$

When $l_1 = l_2$ this gives

$$z = \frac{4l}{5}$$

This agrees with the result obtained on p. 41.

Bending Moment at Support B.—The bending moment influence lines for B are AD, DC, the ordinates being read to the curve.

In this case $z = l_1$ and $z_1 = l_2$

$$\therefore M_B = \frac{l_1 \Sigma W a d}{l_1(l_1 + l_2)} + \frac{l_2 \Sigma W e f}{l_2(l_1 + l_2)}$$

$$= \frac{\sum_A^C W \cdot a d}{(l_1 + l_2)} \qquad (16a)$$

$$\text{Now } a d = y - \frac{\text{BD} \times a_1}{l_1}$$

$$= y - a_1 l_2 = \left(\frac{1}{2}a_1 l_1 + a_1 l_2 - \frac{a_1^3}{2l_1}\right) - a_1 l_2$$

$$= \frac{1}{2}a_1 l_1 - \frac{a_1^3}{2l_1}$$

$\therefore$ When AB is covered, area under AB

$$= -\int_0^{l_1}\left(\frac{1}{2}a_1 l_1 - \frac{a_1^3}{2l_1}\right) d a_1$$

$$= \left(\frac{a_1 l_1^2}{4} - \frac{a_1^4}{8l_1}\right)_0^{l_1}$$

$$= -\frac{l_1^3}{8}$$

Similarly when B C is covered, area under B C

$$= \frac{-l_2^3}{8}$$

∴ when both spans are covered with a load of intensity p we have from equation (16*a*)

$$M_B = \frac{p\,l_1}{l_1(l_1+l_2)} \cdot \frac{-l_1^3}{8} + \frac{p\,l_2}{l_2(l_1+l_2)} \cdot \frac{-l_2^3}{8}$$

$$= \frac{-p\,(l_1^3+l_2^3)}{8\,(l_1+l_2)}$$

This is the familiar result obtained by the Theorem of Three Moments.

Bending Moment Diagram for any Load System.—The simplest procedure for drawing the bending moment diagram for

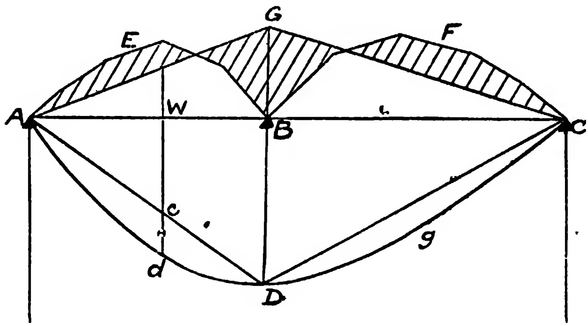


Fig. 28.

any load system in any given position is to first draw the 'free bending moment diagrams' A E B, B F C, Fig. 28, in accordance with the ordinary simple constructions, and draw also the influence curves A d D, D g C.

Multiply each load W by the corresponding ordinate $c\,d$ and add together the results thus obtained and divide by (l_1+l_2) to get M_B; then set up B G equal to the value of M_B thus obtained, and the shaded diagram will give the bending moment diagram required.

The alteration of the shear diagram due to continuity can be obtained readily from the bending moment diagram by correcting the shear base line for the change in slope of the B.M. base line.

*** Application to Continuous Framed Girders.**—The foregoing treatment can be applied to the determination of the stresses in continuous framed girders of constant moment of inertia. Experiments in Germany upon continuous reinforced concrete beams of variable section show that the divergence of the actual support moments obtained from those calculated on the assumption of a constant moment of inertia are practically negligible. We shall indicate in Chapter V., when dealing with the deflections of framed structures, how the reactions in continuous framed girders can be obtained with greater accuracy.

PARALLEL FLANGE GIRDERS.—Take the girder shown in Fig. 29, and suppose that the stresses are required across the bay P Q N M.

To get the stress in the bar P Q we take moments about the point M.

$$\text{Then stress in PQ} = f_{\text{PQ}} = \frac{M_M}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\text{Similarly stress in MN} = f_{\text{MN}} = \frac{M_N}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The influence lines for f_{PQ} and f_{MN} are the same as the bending moment influence lines for M and N, and are therefore given by A U, U C and A R, R C respectively, the ordinates being measured to the curve. (A R is omitted in the figure for clearness.)

To get the stress in M Q we find the shear across the bay and resolve it in the direction M Q. Suppose that a load W is placed on the bay at a distance x from the point N. Then load at

$$\text{N} = W\left(1 - \frac{x}{\text{MN}}\right).$$

Now draw A H parallel to C E and join H R.

Then shear in bay due to unit load

$$= S = R_A - \frac{x}{\text{MN}}$$

Now if the unit load were placed at M, we know from previous reasoning (p. 51) that the shear due to the load would be H m, regarding A E as unity.

∴ We may write

$$\frac{\text{S}\,e}{\text{H}\,m} = \frac{x}{\text{M N}}$$

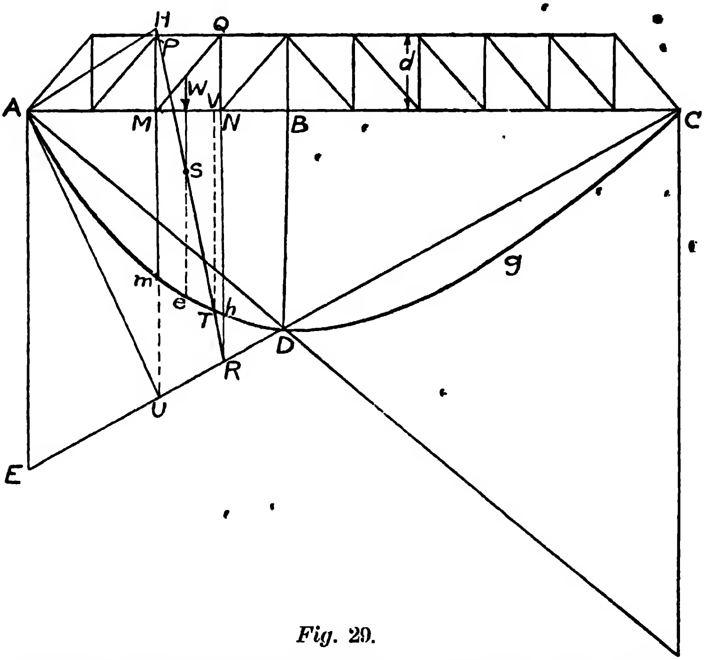


Fig. 29.

Influence Lines for Framed Continuous Girder of two Spans.

This would be quite true if e extended to a line joining m R.

∴ S e gives approximately the shear due to unit load at distance x from N. A H R C is the shear influence line, ordinates being measured from the curve A D C; it is approximate because it is not straight between H and R since the base line is curved.

Uniformly distributed Load.—Let the vertical through the point T, where H R cuts the curve A D C, cut A B in V; then since the

areas A H T and D g C are negative and the area T R D is positive, the maximum negative shear occurs in the bay M N when A V and B C are covered, and the maximum positive shear occurs when V B only is covered.

GIRDER WITH CURVED FLANGE.—In this case, if we consider the section X X, Fig. 30, to get the stress in the upper flange

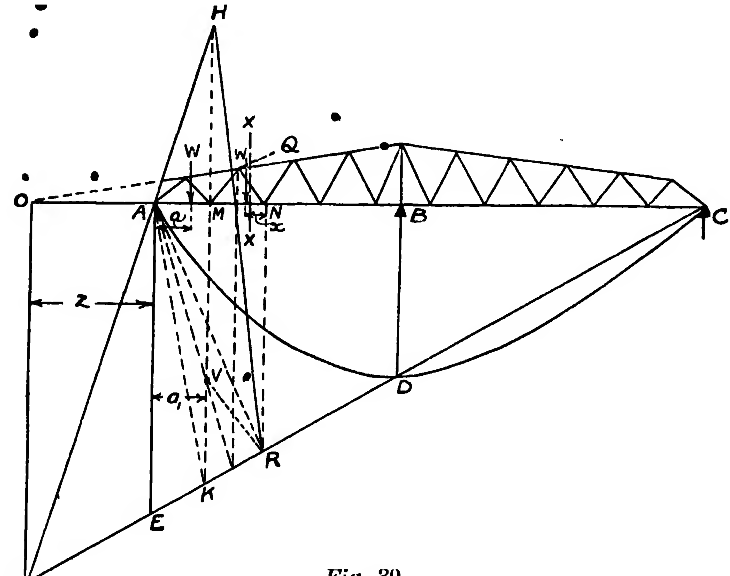


Fig. 30.

Influence Line for Framed Continuous Girder of two Spans.

member we take moments about N; the influence line for this bar therefore is A R C.

For the stress in the lower flange member M N we take moments about the point Q, and for this bar the influence line will be A V R C (straight between V and R).

Now consider the diagonal Q N; to get the stress in this bar we take moments about the point O.

This moment

$$M_o \quad - \left\{ R_A \cdot z - W(z + a) - \frac{W_1 x}{MN}(z + a_1) \right\}$$

Now produce C E to meet the vertical through O in G and join G A, producing it to meet the vertical through M in the point H.

Then by similar reasoning to that employed in the corresponding case of a simply supported beam, it can be shown that G H R C is the influence line for M_O and therefore for the stress in the diagonal Q N.

Continuous Beams of more than two Spans.—Dr. Lea shows in his paper that an extension of the same method is applicable to more than two spans, although the working is rather

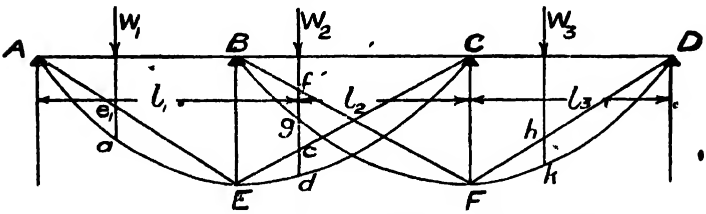


Fig. 31.—*Influence Lines for Continuous Beam of three Spans.*

more complicated. We will just give the results for the determination of the support moments in the case of three spans.

Draw the curves A E C, B F D, Fig. 31, for the first two and second two spans, these curves being the same as for A B, B C and B C, C D considered as two spans only.

$$\text{Then } M_B = G_1(\Sigma W_1 a e_1 + \Sigma W_2 d c) - F(\Sigma W_2 g f + \Sigma W_3 k h)$$

$$\text{and } M_C = F(\Sigma W_1 a e_1 + \Sigma W_2 d c) - G(\Sigma W_2 g f + \Sigma W_3 k h)$$

$$\text{where } F = \frac{\frac{l_2}{2}}{(l_1 + l_2)(l_2 + l_3) - \frac{l_2^2}{4}}$$

$$G = \frac{l_1 + l_2}{(l_1 + l_2)(l_2 + l_3) - \frac{l_2^2}{4}}$$

$$\frac{l_2 + l_3}{(l_1 + l_2)(l_2 + l_3) - \frac{l_2^2}{4}}$$

These quantities F, G, G_1, are constants for any three given spans.

When M_B and M_C have been found in this way, the B. M. diagram is drawn from the free B. M. diagram in the same way as for two spans.

CHAPTER IV

INFLUENCE LINES FOR ARCHES AND SUSPENSION BRIDGES.

We will now consider the application of the method of influence lines to the determination of the stresses in arches and suspension bridges. In so doing we shall make use of some of the results that we shall obtain in Chapters VII., VIII.; this is done in order to keep all of the matter relating to influence lines together, and readers who are not acquainted with such rules are recommended to consult first the later chapters.

THREE-PINNED ARCHES.

Suppose that an isolated load P acts at a point F on an arch with hinges at A, G, C, Fig. 32.

Then the reactions at A and C are made up of vertical components V_A, V_C and the horizontal thrust H.

By taking moments about C we get

$$V_C \cdot L = P\,a\,L$$

$$i.e. \qquad V_C = P\,a \qquad (1)$$

$$\text{Similarly} \qquad V_A = P\,(1 - a) \qquad (2)$$

Again, since the bending moment must be zero at G we get

$$V_C \cdot \frac{L}{2} - H\,r = 0$$

$$\text{or} \qquad H = \frac{V_C\,L}{2\,r} = \frac{P\,a\,L}{2\,r} \qquad (3)$$

For unit load therefore

$$H = \frac{a\,L}{2\,r} \qquad (4)$$

When the load comes on the other side of the point G, the moment at G

$$= V_C \frac{L}{2} - P\left(a - \frac{1}{2}\right) L - Hr = 0$$

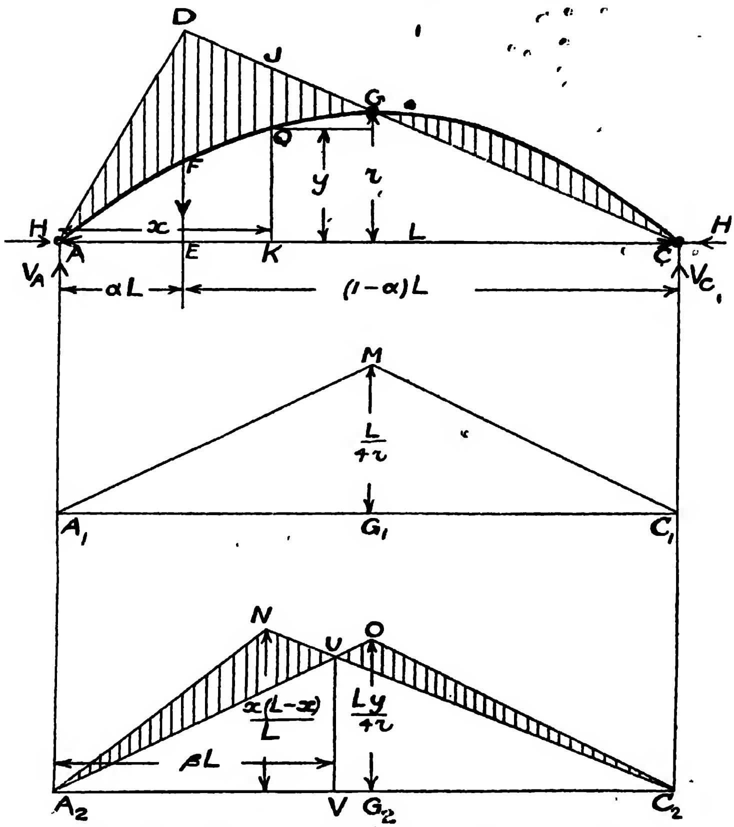


Fig. 32.—Influence Lines for three-pinned Arch.

This gives
$$H = \frac{(1 - a) L}{2r}$$

The influence line, therefore, for the horizontal thrust is a triangle $A_1\ M\ C_1$, the centre ordinate being equal to $\frac{L}{4\,r}$

Bending Moment Influence Line.—Now the bending moment at a point Q between the point F and the end C will be equal to J Q, *i.e.*

$$M_Q = V_C\,(L - x) - H\,.\,y = P\,a\,(L - x) - \frac{a\,L\,y}{2\,r} \;\ldots\ldots\; (5)$$

This is clearly made up of the difference between the bending moment on a freely supported beam of the same span and the moment due to the horizontal thrust.

The influence line for bending moment consists, therefore, of the difference between the two Δs $A_2\ N\ C_2$ and $A_2\ O\ C_2$, the first mentioned being of height $\frac{x\,(L - x)}{L}$ and the second one of height $\frac{L\,y}{4\,r}$., *i.e.*, putting $a = \frac{x}{L}$ and $\frac{1}{2}$ respectively in (5).

Uniformly distributed Loads.—With a uniformly distributed load, the maximum positive bending moment will clearly occur when the length $A_2\ V$ is covered by the load and the maximum negative bending moment will occur when $V\ C_2$ only is covered.

We will now endeavour to calculate the length $A_2\ V = \beta\,L$

$$\text{Now } \frac{U\,V}{A_2\,V} = \frac{O\,G_2}{A_2\,G_2} = \frac{L\,y}{4} \div \frac{L}{2}$$

$$= \frac{y}{2\,r}$$

$$\therefore\ U\,V = \frac{y\,.\,A_2\,V}{2\,r} = \frac{y\,\beta\,L}{2\,r} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6)$$

$$\text{also } \frac{U\,V}{V\,C_2} = \frac{x\,(L - x)}{L} \div (L - x)$$

$$\therefore\ U\,V = \frac{V\,C_2\,.\,x\,(L - x)}{L\,(L - x)} = \frac{L\,(1 - \beta)\,x\,(L - x)}{(L - x)\quad L}$$

$$= (1 - \beta)\,x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (7)$$

$$\therefore\ \frac{y\,\beta\,L}{2\,r} = (1 - \beta)\,x$$

$$\therefore\ \beta\left(x + \frac{y\,\mathrm{L}}{2\,r}\right) = x$$

$$\text{or } \beta = \frac{1}{1 + \dfrac{y\,\mathrm{L}}{2\,r\,x}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (8)$$

Clearly this is usually something less than $\frac{1}{2}$ because $\frac{y\,\mathrm{L}}{2\,r}$ will nearly always be greater than x.

If the intensity of the load is p.

Bending moment at Q when A_2 V is loaded,

$$= \mathrm{M}_{\mathrm{Q}} = p \times \text{area of } \Delta\, \mathrm{A}_2\,\mathrm{N}\,\mathrm{U}$$

$$\text{Now the height of this } \Delta = \frac{x\,(\mathrm{L} - x)}{\mathrm{L}} - \frac{x\,y}{2\,r}$$

$$\therefore\ \mathrm{M}_{\mathrm{Q}} = \frac{p\,\beta\,\mathrm{L}}{2}\left\{\frac{x\,(\mathrm{L} - x)}{\mathrm{L}} - \frac{x\,y}{2\,r}\right\} \quad \dots\dots\dots\dots\dots \quad (9)$$

Parabolic Arch.—Now let the curve of the arch be a parabola. Then we shall have from the property of the parabola

$$\frac{r - y}{r} = \frac{\left(\frac{\mathrm{L}}{2} - x\right)^2}{\left(\frac{\mathrm{L}}{2}\right)^2}$$

$$\textit{i.e. } 1 - \frac{y}{r} = \frac{\dfrac{\mathrm{L}^2}{4} - \mathrm{L}\,x + x^2}{\dfrac{\mathrm{L}^2}{4}}$$

$$= 1 - \frac{4\,(\mathrm{L}\,x - x^2)}{\mathrm{L}^2}$$

$$\therefore\ \frac{y}{r} = \frac{4\,x\,(\mathrm{L} - x)}{\mathrm{L}^2}$$

Then from (8) above

$$\beta = \frac{x}{x + \dfrac{2\,x\,(\mathrm{L} - x)\,.\,\mathrm{L}}{\mathrm{L}^2}}$$

$$= \frac{\mathrm{L}\,x}{\mathrm{L}\,x + 2\,x\,(\mathrm{L} - x)} = \frac{}{3\,\mathrm{L} - 2\,x}$$

$$\therefore \text{ from (9) } \mathrm{M}_{\mathrm{Q}} = \frac{p\,\mathrm{L}^2}{2\,(3\,\mathrm{L} - 2\,x)}\left\{\frac{x\,(\mathrm{L} - x)}{\mathrm{L}} - \frac{2\,x^2\,(\mathrm{L} - x)}{}\right\}$$

$$= \frac{p\,L^2 . x(L - x)}{2(3L - 2x) . L}\left\{1 - \frac{2x}{L}\right\}$$

$$= \frac{p\,x(L - x)(L - 2x)}{2(3L - 2x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Now this bending moment will be a maximum when

$$\frac{d\,M_Q}{dx} = 0 \quad \textit{i.e.}\text{ when } \frac{d\left\{\dfrac{p\,x(L^2 - 3Lx + 2x^2)}{2(3L - 2x)}\right\}}{dx} = 0$$

i.e. when

$$\frac{p}{2(3L - 2x)^2}\left\{(3L - 2x)(L^2 - 6Lx + 6x^2) + 2x\,(L^2 - 3Lx + 2x^2)\right\} = 0$$

$$\textit{i.e.}\text{ when } 3L^3 - 18L^2x + 24Lx^2 - 8x^3 = 0 \ldots\ldots (11)$$

A solution of this equation by plotting gives

$$x = \cdot225\,L \text{ approx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$\text{This gives } \beta = \frac{L}{3L - \cdot45\,L}$$

$$= \cdot392 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Putting these values into the value for M_Q we get maximum bending moment $= \cdot0188\,p\,L^2$

$$= \frac{p\,L^2}{53} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

This occurs at a point just before $\frac{1}{4}$ span ($\cdot225$ L), when the load extends from one end to about $\frac{2}{5}$ span ($\cdot392$ L).

This result agrees with the value that can be obtained by a different method either for a three-hinged arch or the equivalent case of a stiffening girder for a suspension bridge pin-jointed at the centre.*

Shear and Thrust Influence Lines.—Now consider the shear and thrust at point Q.

Shear.—If θ is the inclination with the horizontal of the arch at the point Q, Fig. 33, then the shear at the point Q will be the resultant force normal to the arch at that point.

* A, p. 368.

If the load W is between A and Q, the shear at Q

$$= S_Q = V_C \cos\theta + H \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

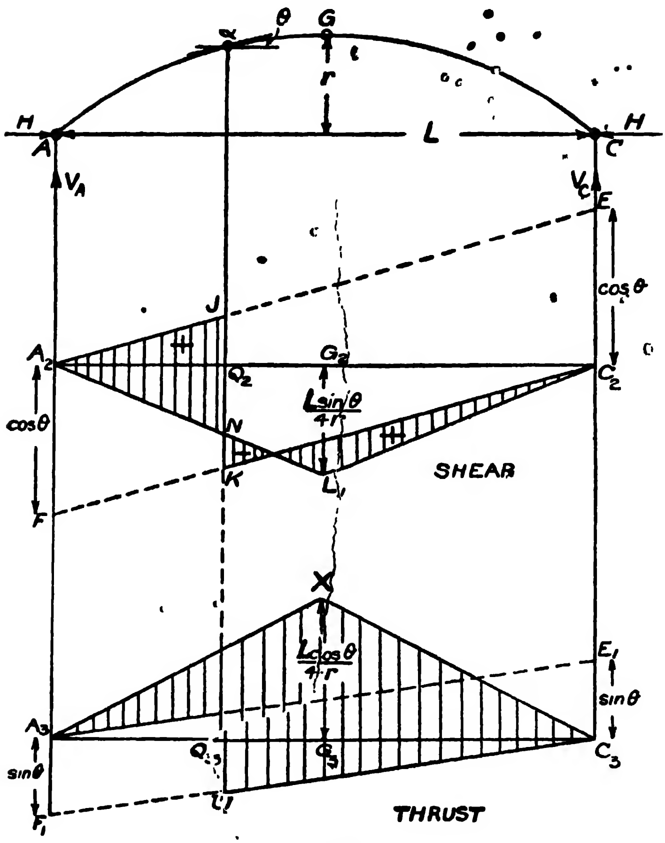


Fig. 33.

Shear and Thrust Influence Lines for three-pinned Arch.

If the load is between Q and C

$$S_Q = V_C \cos\theta - W\cos\theta + H\sin\theta = -V_A \cos\theta + H \sin\theta \ldots(2)$$

These expressions for unit load will be the same as for a simply supported beam with a load equal to cos θ with H sin θ added.

We get, therefore, the shear influence line as follows :—

Set up and down respectively lengths C_2 E and A_2 F each equal to cos θ and join A_2 E and C_2 F, intersecting the vertical through Q in J and K ; at the centre set down a length $G_2 L_1$ equal to $\frac{L \sin \theta}{4 r}$ the ordinates of the Δ $A_2 L_1 C_2$ representing H sin θ, then the portion shown shaded is the shear influence line for the arch.

Thrust.—The thrust at Q will be the resultant force tangential to the arch at the point.

For the load between A and Q, the thrust at Q

$$= T_Q = H \cos \theta - V_C \sin \theta \quad \ldots\ldots\ldots\ldots\ldots \text{(3)}$$

If the load is between Q and C

$$T_Q = H \cos \theta - V_C \sin \theta + W \sin \theta$$
$$= H \cos \theta + V_A \sin \theta \quad \ldots\ldots\ldots\ldots\ldots \text{(4)}$$

By similar reasoning to that for the shear, we get the following construction for the thrust influence line :—

Make $C_3 E_1$ and $A_3 F_1$ equal to sin θ and join across to meet the vertical through Q. At the centre set up G_3 X equal to $\frac{L \cos \theta}{4 r}$; then the portion shown shaded is the thrust influence line, from which it is clear that in nearly every case the maximum thrust with a uniformly distributed load is obtained when the whole span is covered.

Three-pinned Framed Spandril Arches.—In the case of the spandril arch with three hinges, Figs. 34 and 35, we can find the stresses in any bay such as D E F B by the method of moments.

Stress in D E.—To obtain the stress in the top horizontal member D E we take moments about the point B.

Then we have :

$$\text{Stress in D E} = f \quad \frac{\text{moment about B of forces to left of x x}}{\text{B D}}$$

$$\frac{M_B}{\text{B D}}$$

As B D is a constant, the influence line for the stress in D E will be the same as that for the bending moment at B, and will thus be given by the difference of the triangles A_2 N C_2 and A_2 O C_2 as previously explained, + indicating compression stress and – indicating tension.

Stress in B F.—To obtain the stress in B F we must take moments about the point E.

Then

$$\text{Stress in B F} = f_{BF} = \frac{\text{moment about E of forces to one side of X X}}{u}$$

$$= \frac{M_E}{u}$$

Now M_E for a load between E and the left-hand end

$$= V_C (L - z) - H h$$

i.e. $\qquad M_E$ = free bending moment – $H h$

If, therefore, we set up F_1 K equal to $\frac{z (L - z)}{L}$ and join K A_1 and K C_1 we shall get the influence line for the free bending moment; then set up G_1 J equal to $\frac{L h}{4 r}$ and join J A_1 and J C_1, thus giving the influence line for the quantity $H h$. The influence line for the stress in B F is therefore given by the difference of the two triangles A_1 K C_1 and A J C_1.

If, for example, a load W is placed as shown in Fig. 34, the stress in B F will be equal to $\frac{W m}{u}$.

Stress in Diagonal B E.—To obtain the stress in B E we have to take moments about the point Y where E D and F B meet when produced.

Then

$$\text{Stress in B E} = f_{BE} = \frac{\text{moment about Y of forces to one side of X X}}{v}$$

$$= \frac{M_Y}{v}$$

If the load is to the left of E

$$M_Y = V_C \cdot e - H h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

If the load is to the right of D

$$M_Y = V_A (L - e) - H h \quad \ldots\ldots\ldots\ldots \quad (2)$$

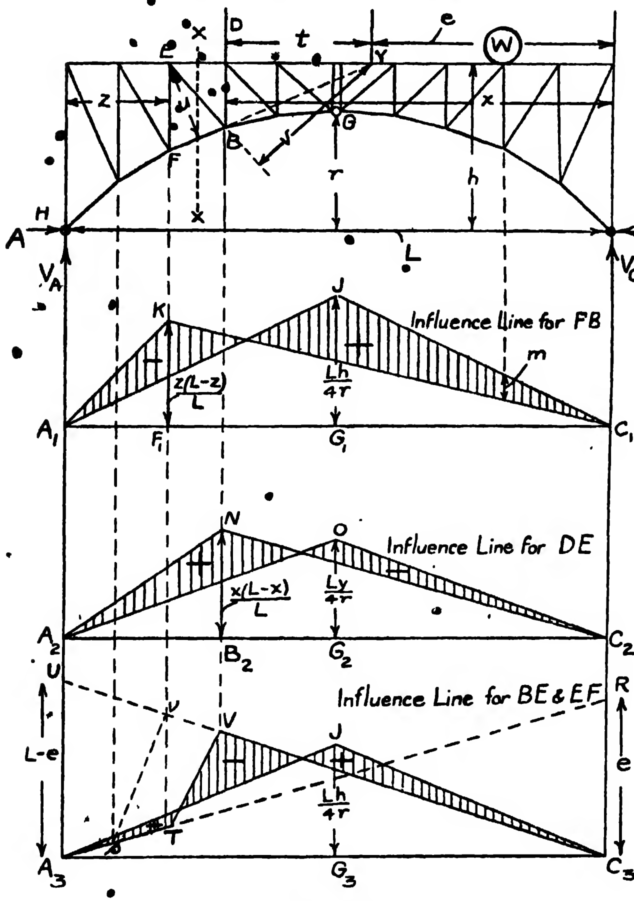


Fig. 34.

Influence Lines for three-pinned Spandril Arch.

Considering first the first term in each of these two expressions, if α L is the distance of the unit load from A, $V_C = \alpha$ and $V_A = (1 - \alpha)$. (See p. 61.)

In equation (1), therefore, the first term $V_C\, e$ will be given by a straight line, the ordinate at the end C of which would be equal to e (*i.e.*, putting $a = 1$).

Therefore set up C_3 R to represent the length e and draw A_3 R up to the point T where it cuts the vertical through E.

In equation (2) the first term V_A (L $- e$) will similarly be given by setting up A_3 U to represent (L $- e$) and joining C_3 U up to the point V where it cuts the vertical through D because $V_A = (1 - a)$ and this equals 1 when $a = 0$ and 0 when $a = 1$.

By the rule that an influence line is straight between points of load, we get by joining V T the complete influence line A_3 T V C_3 for the first portion of M_Y.

To allow for the term H y, set up G_3 J to represent $\frac{L\,h}{4\,r}$ and join A_3 J and C_3 J, the complete influence line then coming as shown shaded in Fig. 34.

Stress in Vertical E F.—To obtain the stress in the vertical E F we take moments about the same point Y and shall get:

$$\text{Stress in E F} = f_{EF} = \frac{M_Y}{t}$$

In this case however the section line comes to the left of E, so that the influence line for E F is given by the dotted line $s\,v$, being otherwise the same as for B E except that the signs should be reversed.

If the point Y comes outside the span, the distance e is regarded as negative in drawing the influence line.

NUMERICAL EXAMPLE OF THREE-PINNED SPANDRIL ARCH.—*A three-pinned arch of* 150 *ft. span and rise* 30 *ft. is divided into ten equal bays and is* 6 *ft. deep at the centre* (*Fig.* 35). *The top chord is horizontal and the lower chord is parabolic. Find the maximum stresses in the members* E D, F B, *and* B E *when loads of* 15 *tons each may occur at the points a, b, c, and d, &c.*

To draw the influence lines first set up G_1 J at the centre equal to $\frac{L\,h}{4\,r} = \frac{150 \times 36}{4 \times 30} = 45$ ft. and join J to A_1 and C_1.

Now set up F_1 K equal to $\frac{z\,(L - z)}{L} = \frac{30 \times 120}{150} = 24$ ft. and join K to A_1 and C_1

Produce F B to meet the top chord at Y; then Y on measurement

comes 78 ft. from the right-hand end. Set up C_1 R = 78 and A_1 U = 150 − 78 = 72 and join across to A_1 and C_1 as far as the verticals through F and B respectively and join T V.

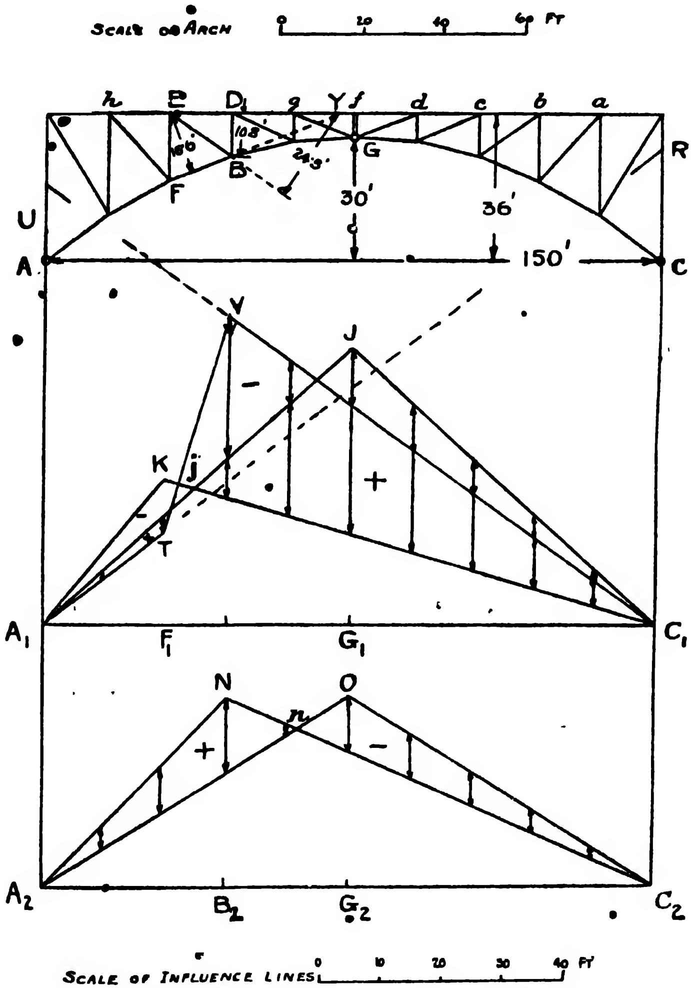


Fig. 35.—*Three-pinned Spandril Arch.*

Then A_1 K C_1 and A_1 T V C_1 are the influence lines for stresses in F B, B E respectively, ordinates being measured from the line A_1 J C_1.

Maximum Stress in F B.—It is clear that the maximum stress in F B occurs when the loads extend from a to D, because beyond that point the influence line becomes negative; the sum of the ordinates between the lines j C_1 and j J C_1 comes equal to 114·5 and the distance u = 15·6. The load at each point is 15 tons

$$\therefore \text{ maximum stress in F B} = \frac{+114{\cdot}5 \times 15}{15{\cdot}6} = +110 \text{ tons nearly.}$$

$$\therefore \textit{max.} \int_{\text{F B}} \textit{(compression)} = 110 \textit{ tons nearly.}$$

Maximum Stress in B E.—The influence line for B E has two positive portions which give a maximum stress when the points h, E, f, d, c, b, a are loaded; the sum of the positive ordinates between the lines A_1 T V C_1 and A_1 J C_1 comes equal to + 30·4 ft. and v = 24·5 ft.

$\therefore$ Maximum compression stress in B E $= \dfrac{30{\cdot}4 \times 15}{24{\cdot}5} = 18{\cdot}6$ tons nearly.

$$\int_{\text{B E}} \textit{(compression)} = 18{\cdot}6 \textit{ tons.}$$

It is not obvious from the figure whether or not this is numerically greater than the maximum tensile stress, which occurs clearly when the points D g are loaded.

The sum of the two ordinates of the influence line = − 31.

$$\therefore \text{ Maximum tensile stress in B E} = \frac{31 \times 15}{24{\cdot}5} = 19 \text{ tons nearly.}$$

$$\therefore \textit{Max.} f_{\text{BE}} \textit{ (tension)} = 19 \textit{ tons.}$$

Maximum Stress in D E.—To draw the influence line for D E, set up at the centre of a base A_2 C_2 a length G_2 O $= \dfrac{L\,y}{4\,r}$

$$= \frac{150 \times 25{\cdot}2}{4 \times 30} = 31{\cdot}5 \text{ ft., and join O to } A_2 \text{ and } C_2$$

Next set up B_2 N $= \dfrac{x\,(L - x)}{L} = \dfrac{45 \times 105}{150} = 31{\cdot}5$ ft. and join N to A_2 and C_2.

Then the influence line for D E is the difference between A_2 N C_2 and A_2 O C_2.

Considering first the tensile stresses, the maximum stress is clearly obtained when the points f, d, c, b, a are loaded.

The sum of the ordinates for the portion n O C_2 comes equal to 27 ft.

$$\therefore \text{ Maximum tensile stress in D E} = \frac{27 \times 15}{10{\cdot}8} = 37{\cdot}5 \text{ tons.}$$

$$\therefore \textit{Max.} f_{\text{DE}} \textit{ (tension)} = 37{\cdot}5 \textit{ tons.}$$

To get the maximum compression we add the ordinates of the portion $A_2 N n$. This comes the same as before, viz., 27 ft.

$$\therefore \textit{Max. } f_{DE} \textit{ (compression)} = 37{\cdot}5 \textit{ tons.}$$

TWO-PINNED ARCHES.

Bending Moment Influence Line.—We will assume for the present that we are able to calculate the horizontal thrust for a unit load at any point F, Fig. 36, on the arch; it is proved on p. 129 that for a parabolic arch with a certain variation of cross section, the value of H for a load $P = \frac{5}{8}\frac{P\,L\,a}{r}(1 - a)(1 + a - a^2)$.

By putting in various values for a and taking $P = 1$ and plotting the results we can get an influence line $A_1\ G_1\ C_1$ for H, the centre ordinate (where $a = \frac{1}{2}$) coming $\frac{25}{128}\frac{L}{r} = \frac{{\cdot}195\,L}{r}$

Now the bending moment M at Q

$$= M_Q = \text{free bending moment} - H\,y.$$

$$\therefore \frac{M_Q}{y} = \frac{\textit{free bending moment}}{y} - H.$$

It will be much more convenient to draw influence lines for $\frac{M_Q}{y}$ than for M_Q, because we shall then have to draw only one H curve for various positions of Q.

We therefore set up $K_1\,N = \frac{x\,(L - x)}{L\,y}$ and join N to A_1 and C_1 and the shaded curve then gives the influence line for $\frac{M_Q}{y}$

For arches which are not parabolic we must calculate values of H for various positions of the load by summation if no formula is available.

Shear and Thrust Influence Lines.—The shear and thrust influence lines are very similar to those for the three-pinned arch and will be followed without further description from Fig. 36. To save drawing a number of curves for H sin θ and H cos θ, an artifice similar to that employed for the B.M. influence line can be employed, viz., set out instead of H sin θ and H cos θ curves of

H in each case, and instead of cos θ and sin θ set up cot θ and tan θ respectively; the true values of the shear and thrust are then found by multiplying the ordinates of the influence lines by sin θ and cos θ respectively.

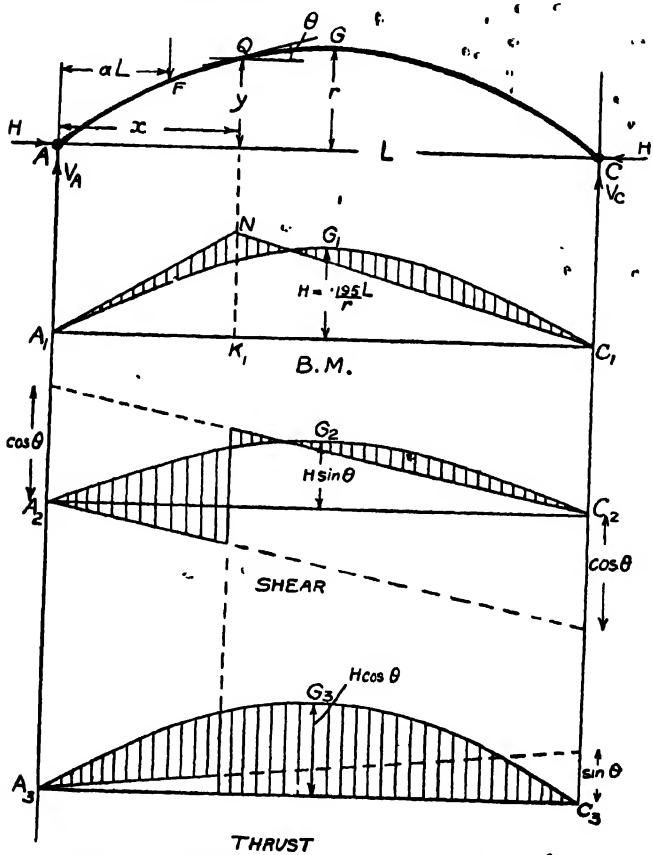


Fig. 36.—Influence Lines for two-pinned Arch.

Two-pinned Spandril Arches.—The influence lines for two-pinned spandril arches are the same as for the three-pinned case except that instead of a triangle for the moment of H we

have a curve as in the previous example, the values of H being calculated as explained on p. 148; Fig. 37 shows the influence lines for this case and will be followed without further

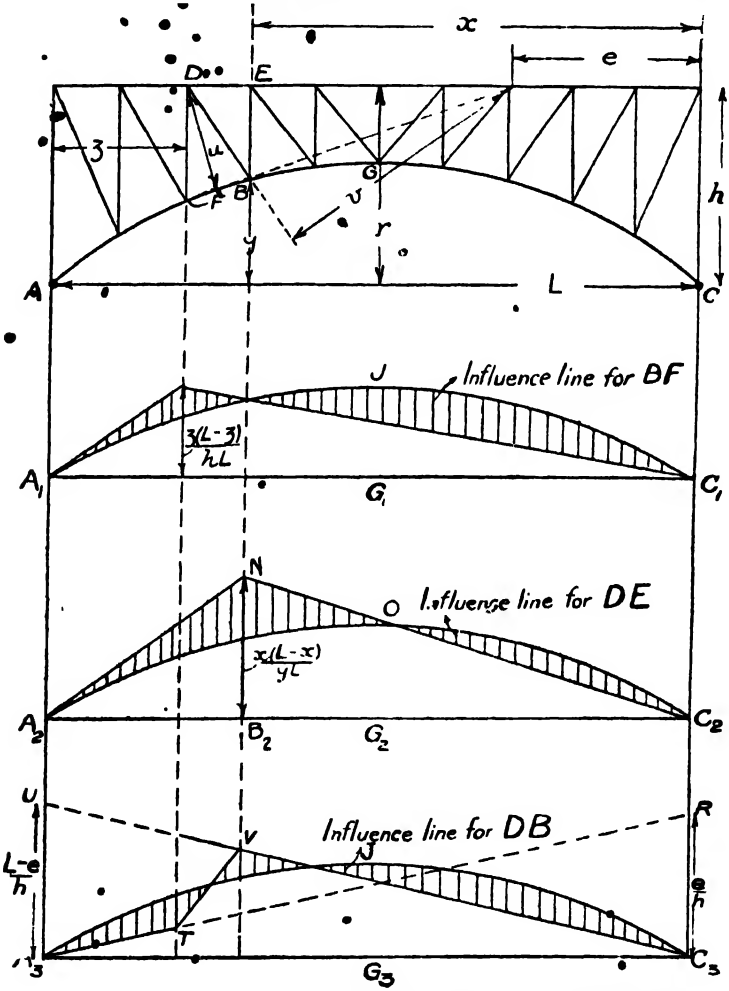


Fig. 37.—Influence Lines for two-pinned Spandril Arches.

description except to note that since the curves $A_1 J C_1$, $A_1 J C_3$, and $A_2 O C_2$ are for H, to get the stress in DE the ordinates of the influence line have to be multiplied by $\frac{y}{BE}$; for stress in FB by $\frac{h}{u}$; and for stress in DG by $\frac{h}{v}$.

Doubly built-in or hingeless Arches.—The calculations involved in this form of arch are considerably more troublesome than for two-pinned arches. Influence lines can however, be drawn and the reader will find some information upon the subject in § 163 of Johnson, Bryan, and Turneare's *Modern Framed Structures*, Vol. II. (Wiley & Sons).

*STIFFENED SUSPENSION BRIDGES.

Girders hinged at Centres and Ends.—The bending moment influence line for this case is exactly the same as for the three-pinned arch.

SHEAR INFLUENCE LINE.—The shearing force at any point of a stiffening girder will be equal to shearing force for a simply supported beam diminished by the vertical component of the tension in the cable.

Consider a point Q at distance x from one end A_1 of the stiffening girder (Fig. 38).

The vertical component of the tension in the cable is equal to H tan θ, where θ is the inclination of the cable to the horizontal at the point Q_1, so that if S_Q is the shear at Q and s_Q is that which would occur for a simply supported beam we shall have

$$S_Q = s_Q - H \tan \theta \quad \text{.......................(1)}$$

$$\text{Now} \quad \tan \theta = \frac{2\,NG}{GQ}$$

because by the property of the parabola NG = QQ′

Also from p. 64.

$$\frac{d_Q}{d} = \frac{4\,x\,(L - x)}{L^2}$$

$$\therefore \quad 1 - \frac{d_Q}{d} = 1 - \frac{4\,x\,(L - x)}{L^2}$$

$$i.e. \quad \frac{d - d_Q}{d} = \frac{NG}{d} = 1 - \frac{4x(L - x)}{L^2}.$$

$$\therefore \tan\theta = \frac{2\,NG}{GQ} = \frac{2\,NG}{\left(\frac{L}{2} - x\right)} = \frac{2d\left(1 - \frac{4x(L - x)}{L^2}\right)}{\left(\frac{L}{2} - x\right)}$$

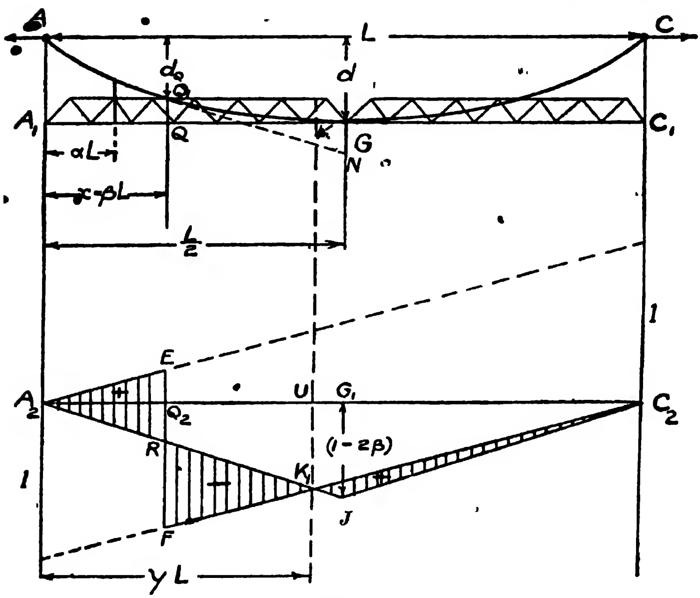


Fig. 38.
Shear Influence Lines for Stiffened Suspension Bridges.

$$= \frac{2d(L^2 - 4xL + 4x^2)}{L^2\left(\frac{L}{2} - x\right)}$$

$$= \frac{2d(L - 2x)^2}{\frac{L^2}{2}\left(L - 2x\right)}$$

$$= \frac{4d(L - 2x)}{L^2}.$$

$$= \frac{4\,d\,(1 - 2\,\beta)}{L} \quad \dots\dots\dots\dots (2)$$

From equation (4) p. 61, for unit load,

$$H = \frac{a\,L}{2\,d}$$

$$\therefore \quad S_Q = s_Q - \frac{a\,L}{2\,d} \cdot \frac{4\,d}{L}\left(1 - 2\,\beta\right)$$

$$= s_Q - 2\,a\,(1 - 2\,\beta), \text{ which holds up to G.}$$

The second term is proportional to a, so that the influence line for the H term is a Δ of height $(1 - 2\,\beta)$ at the centre.

To draw the complete influence line for shear therefore we proceed as follows:—

First draw the shear influence line A_2 E F C_2 for a simply supported beam (*cf.* Fig. 1); at the centre G_1 set down a length equal to $(1 - 2\,\beta)$, then the difference shown shaded gives influence line required.

It may be noted that if $\beta < \frac{1}{4}$, C_2 F comes above the Δ, if $\beta = \frac{1}{4}$ they coincide, and if $\beta > \frac{1}{4}$, C_2 F comes below it.

Maximum Shear for Uniform Loading.—Case I. $\beta < \frac{1}{4}$.—*Negative Shear.*—It is clear from the influence diagram that the maximum negative shear for $\beta < \frac{1}{4}$ is obtained when the length Q K is loaded.

To calculate this maximum negative shear we have to obtain an expression for the area of the Δ F R K_1.

$$\text{Now this area} = \Delta = \frac{1}{2}\,R\,F \,.\, Q_2\,U = \frac{1}{2}\,R\,F\,(\gamma - \beta)\,L$$

$$\text{Now} \quad Q_2\,F = 1 - \beta$$

$$\text{and} \quad \frac{Q_2\,R}{(1 - 2\,\beta)} = \frac{\beta\,L}{\frac{L}{2}}$$

$$\textit{i.e.} \quad Q_2\,R = 2\,\beta\,(1 - 2\,\beta) \quad \dots\dots\dots\dots (2a)$$

$$\therefore \quad R\,F = (1 - \beta) - 2\beta\,(1 - 2\,\beta)$$

$$= 1 - 3\,\beta + 4\,\beta^2$$

$\therefore$ We have max. $S_Q = \frac{1}{2} p L (\gamma - \beta) \left\{ 1 - 3\beta + 4\beta^2 \right\}$ ... (3)

We now have to determine γ in terms of β.

We have $\dfrac{\gamma L}{\frac{L}{2}} = \dfrac{UK_1}{(1 - 2\beta)}$

$\therefore UK_1 = 2\gamma(1 - 2\beta)$ .... (4)

Again $\dfrac{UK_1}{Q_2 F} = \dfrac{C_0 U}{C_2 Q_2} = \dfrac{1 - \gamma}{1 - \beta}$

$\therefore UK_1 = \dfrac{(1 - \gamma)\, Q_2 F}{1 - \beta} = (1 - \gamma)$ ....(5)

$\therefore$ Combining (4) and (5) $1 - \gamma = 2\gamma(1 - 2\beta)$

or $\gamma = \dfrac{1}{3 - 4\beta}$ ....(6)

$\therefore \gamma - \beta = \dfrac{1}{3 - 4\beta} - \beta = \dfrac{1 - 3\beta + 4\beta^2}{(3 - 4\beta)}$

Putting this result into equation (3) above we get

Max. negative $- S_Q = \dfrac{pL}{2} \dfrac{(1 - 3\beta + 4\beta^2)^2}{(3 - 4\beta)}$ ....(7)

The extreme values for β in this expression are o and $\frac{1}{4}$.

For $\beta = 0$, $S_Q = \frac{pL}{6}$, this being the maximum positive shear at the end; in this case $\gamma = \frac{1}{3}$.

For $\beta = \frac{1}{4}$, $S_Q = \frac{pL}{16}$, this being the maximum positive shear at $\frac{1}{4}$ span; in this case $\gamma = \frac{1}{2}$.

Positive Shear.—It is clear from Fig. 38 that the maximum negative shear occurs when the portions $A_1 Q$ and $K C_1$ are loaded.

We shall now prove that the sum of the areas of the Δs below $K C_1$ and $A_1 Q \times p$ which is equal to the maximum positive shear is numerically equal to the maximum negative shear.

Now area of Δ below $K C_1$

$$= \frac{1}{2}\left(1 - 2\beta - \frac{1}{2}\right)(L - \gamma L)$$

$$= \frac{1}{2}\left(1 - 2\beta - \frac{1}{2}\right)(1 - \gamma)\,L$$

$$= \frac{L}{2}\left(\frac{1}{2} - 2\beta\right)\left\{1 - \frac{1}{(3 - 4\beta)}\right\}, \text{ by equation (6)}$$

$$= \frac{L}{2}\left(\frac{1}{2} - 2\beta\right)\left(\frac{2 - 4\beta}{3 - 4\beta}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Area of Δ below $A_1\,Q = \frac{1}{2} A_2 Q_2 \,.\, \overline{ER}$

$$= \frac{1}{2}\beta L\,(EQ_2 + Q_2R)$$

$$= \frac{1}{2}\beta L\left\{\beta + 2(1 - 2\beta)\beta\right\}, \text{ by equation } (2a)$$

$$= \frac{1}{2}\beta^2 L\left\{3 - 4\beta\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$\therefore\ p \times$ sum of areas of Δs $K_1\,J\,C_2$ and $A_2\,E\,R$ = maximum + S_Q

$$= p\,\frac{L}{2}\left\{\frac{(\frac{1}{2} - 2\beta)(2 - 4\beta)}{(3 - 4\beta)} + \beta^2(3 - 4\beta)\right\}$$

$$= \frac{p\,L}{2(3 - 4\beta)}\left\{1 - 6\beta + 8\beta^2 + \beta^2(3 - 4\beta)^2\right\}$$

$$= \frac{p\,L}{2(3 - 4\beta)}\left\{1 - 2\beta(3 - 4\beta) + \beta^2(3 - \beta 4)^2\right\}$$

$$= \frac{p\,L}{2(3 - 4\beta)}(1 - 3\beta + 4\beta^2)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

This is the same numerically as the previous result (7).

Case II. $\beta > \frac{1}{4}$.—The shear influence line for this case is as shown in Fig. 39.

Negative Shear.—The maximum negative shear clearly occurs when the load extends from C_1 to Q.

The negative shear area = shaded trapezium + triangle.

Now area of trapezium

$$= \frac{Q_2\,G_1}{2}\left\{(1 - 3\beta + 4\beta^2) + \frac{1}{2} - (1 - 2\beta)\right\}$$

[because $R\,F$, Fig. 38, $= 1 - 3\beta + 4\beta^2$ (p. 78) and $G_1\,V = \frac{1}{2}$]

$$= \frac{L}{2}\left(\frac{1}{2} - \beta\right)\left\{\frac{1}{2} - \beta + 4\beta^2\right\}$$

And area of $\Delta = \frac{1}{2} \; \frac{L}{2}\left\{\frac{1}{2} - (1 - 2\beta)\right\}$

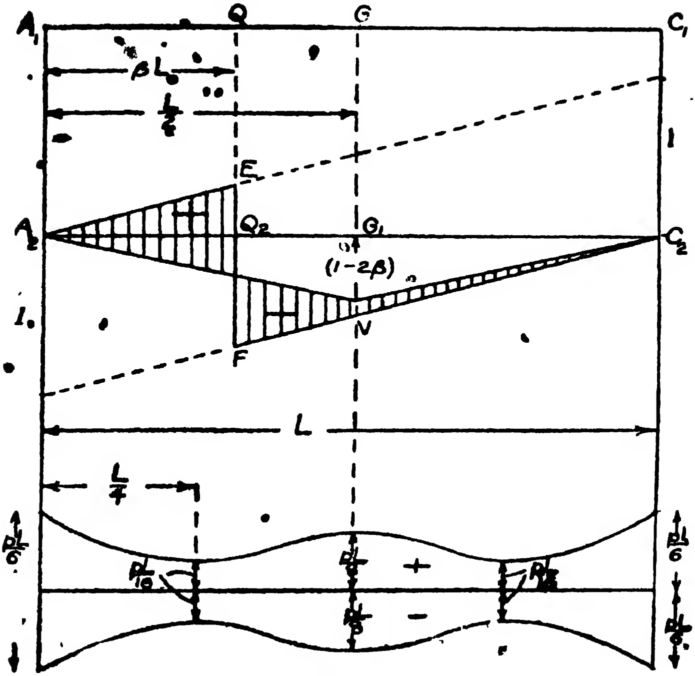


Fig. 39.

Shear Influence Lines for Stiffened Suspension Bridges.

∴ Negative shear area

$$= \frac{L}{2}\left(\frac{1}{2} - \beta\right)\left\{\frac{1}{2} - \beta + 4\beta^2\right\} - \frac{L}{4}\left(\frac{1}{2} - 2\beta\right)$$

$$= \frac{L}{2}\left\{\left(\frac{1}{2} - \beta\right)\left(\frac{1}{2} - \beta + 4\beta^2\right) + \beta - \frac{1}{4}\right\}$$

$$= \frac{L\beta^2}{2}(3 - 4\beta)$$

∴ Max. negative $S_Q = \frac{pL\beta^2}{2}(3 - 4\beta)$ ....(11)

This is a maximum when $\frac{d\,\beta^2(3 - 4\beta)}{d\beta} = 0$

i.e. $6\beta^2 - 12\beta^3 = 0$

or $\beta = \frac{1}{2}$

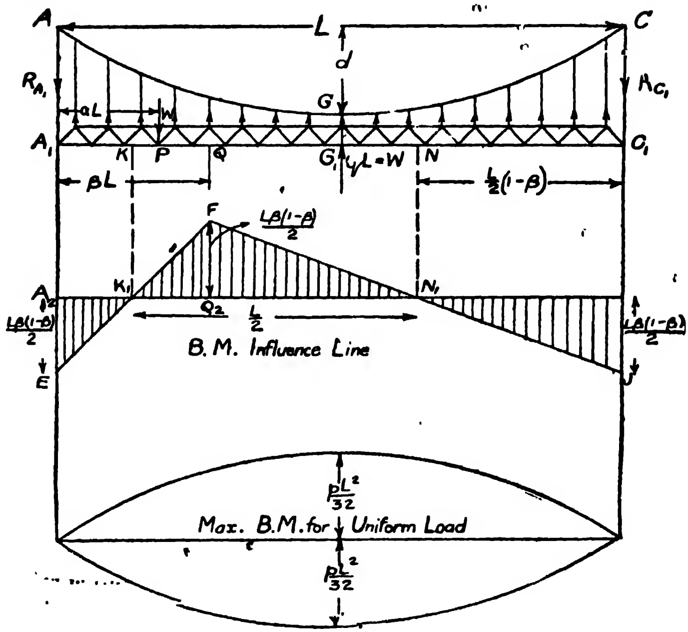


Fig. 40.

Influence Lines for Stiffened Suspension Bridges.

Then max. negative $S_Q = \frac{p\,L}{8}$

Positive Shear.—The maximum positive shear occurs when the load extends from A_1 to Q.

Then positive shear area = area of positive Δ.

$$= \frac{\beta^2 L}{2}(3 - 4\beta) \text{ from equation (9).}$$

$$\therefore \text{ Max. positive } S_Q = \frac{p\,L\,\beta^2}{2}(3 - 4\beta) \qquad (12)$$

This is numerically the same as the negative shear and so we get the curve of maximum shears as shown at the bottom of Fig. 39.

Girders hinged at Ends only.—The assumption that is usually made in the approximate treatment of this case is that the cable retains the form of the parabola and that therefore there is a uniform upward pull from the cable on the stiffening girder.

A more accurate approximate method is to treat the whole structure as a redundant frame; this treatment and also an exact method of calculation is very fully treated in Johnson, Bryan & Turneare's *Modern Framed Structures*, Vol. II.

Adopting the usual treatment and taking a load W at the point P, Fig. 40, the upward pull q per foot run $= \frac{W}{L}$; the resultant upward pull W acting at the centre.

These upward and downward forces W form a couple or moment equal to $W \, PG_1 = WL\left(\frac{1}{2} - a\right)$ which is equal to the moment of the couples formed by the reactions.

$$\therefore \; -R_{C_1} = +R_{A_1} = W\left(\frac{1}{2} - a\right) \qquad \ldots\ldots (1)$$

$\therefore$ Bending moment at Q (treating according to the usual rule anticlockwise moments to the right as positive, *i.e.* as positive bending moments those which will cause the bottom flange to be in tension)

$$\begin{aligned} &= M_Q = -R_{C_1} . QC_1 + q . QC_1 . \frac{QC_1}{2} \\ &= -QC_1\left\{R_{C_1} - \frac{W}{L} . \frac{QC_1}{2}\right\} \\ &= -L(1-\beta)\left\{W\left(\frac{1}{2} - a\right) - \frac{W.(1-\beta)}{2}\right\} \\ &= -W.L(1-\beta)\left\{\left(\frac{1}{2} - a\right) - \frac{(1-\beta)}{2}\right\} \\ &= -\frac{WL}{2}(1-\beta)(\beta - 2a) \qquad \ldots\ldots (2) \end{aligned}$$

Bending Moment Influence Line.—For unit load between Q and A_1, $M_Q = -\frac{L}{2}\left(1 - \beta\right)\left(\beta - 2a\right)$.

For a given position of Q, M_Q varies as a and so the bending moment influence line will be a straight line. The ordinate at the end A_1 will be given by $a = 0$, and will be equal to $\frac{-L\beta}{2}\left(1 - \beta\right)$, and at the point Q, where $a = \beta$, it will be equal to $+\frac{L\beta}{2}\left(1 - \beta\right)$. When the load is beyond Q, $a > \beta$ and M_Q for unit load will be equal to the previous value *minus* $1 \times L(a - \beta)$.

$$\therefore M_Q = \frac{-L}{2}\left(1 - \beta\right)\left(\beta - 2a\right) - L\left(a - \beta\right)$$

$$\frac{-L}{2}\left\{\beta - 2a + 2a\beta - \beta^2 + 2a - 2\beta\right\}$$

$$= \frac{-L\beta}{2}\left\{2a - \beta - 1\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The extreme value of M_Q for $a = 1$ comes equal to $\frac{L\beta(1 - \beta)}{2}$, and so we get the bending moment influence line as shown in the figure.

The maximum positive and negative bending moments occur at the centre, and are equal to $\frac{WL}{8}$.

Maximum Bending Moment with Rolling Uniform Load.—It is clear from the bending moment influence line that the maximum positive bending moment at Q for a rolling uniform load will occur when K N is covered, and the maximum negative bending moment when A_1 K and N C_1 are covered, the numerical values being equal.

$$\therefore \text{Max. } M_Q = p \times \text{area of } \Delta\ K_1 F N_1$$

$$= \frac{pL}{4} \cdot \frac{L\beta(1 - \beta)}{2}$$

$$= \frac{pL^2}{8}\beta(1 - \beta) \quad \ldots\ldots\ldots\ldots \quad (4)$$

This has a maximum value when $\frac{d\beta(1 - \beta)}{d\beta} = 0$, *i.e.* $\beta = \frac{1}{2}$, then we get for the maximum possible bending moment the value

$$\text{Max. } M_Q = \frac{pL^2}{32} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

If the rolling load is of indefinite length, the maximum positive bending movement at Q will occur when A_1 N is covered, and the maximum negative when N C_1 is covered, the numerical values being equal.

$$\text{Then max. } M_Q = p \times \text{area of } \Delta \text{ below N } C_1$$

$$= \frac{p}{2} \cdot \frac{L}{2}\left(1 - \beta\right) \cdot \frac{L\beta(1 - \beta)}{2}$$

$$= \frac{p L^2}{8} \beta (1 - \beta)^2 \qquad \ldots\ldots\ldots\ldots \quad (6)$$

This is a maximum when $\frac{d \beta (1 - \beta)^2}{d \beta} = 0$

$$i.e. \frac{d(\beta - 2\beta^2 + \beta^3)}{d\beta} = 1 - 4\beta + 3\beta^2 = 0$$

This gives $\beta = \frac{1}{3}$ for which we get

$$\text{Max. } M_Q = \frac{p L^2}{54} \qquad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

Some writers have referred to the value $\frac{p L^2}{54}$ as erroneous; it is, however, correct for loads of indefinite length, or, in fact, of length greater than one-third of the span.

Shear Influence Line.—When the load is between A_1 and Q the shear at Q

$$= S_Q = R_{C_1} - q \cdot QC_1$$

$$= W\left(\frac{1}{2} - a\right) - \frac{W}{L} \cdot L(1 - \beta)$$

$$= W\left(\beta - a - \frac{1}{2}\right) \qquad \ldots\ldots\ldots\ldots \quad (8)$$

$$\therefore \text{ For unit load } S_Q = \left(\beta - a - \frac{1}{2}\right)$$

This is a linear relation, so that the shear influence line will be a straight line.

At the end A_1, where $a = 0$, the ordinate of the influence line will be equal to $\left(\beta - \frac{1}{2}\right)$, and at the point Q where $a = \beta$, it will be equal to $-\frac{1}{2}$.

Now take the load between Q and C_1.

$$\text{Then } S_Q = R_{C_1} - q \cdot QC_1 + W$$

$$= W\left(\beta - \alpha + \frac{1}{2}\right) \quad \ldots\ldots\ldots \quad (9)$$

$$\therefore \text{ for unit load } \quad S_Q = \left(\beta - \alpha + \frac{1}{2}\right)$$

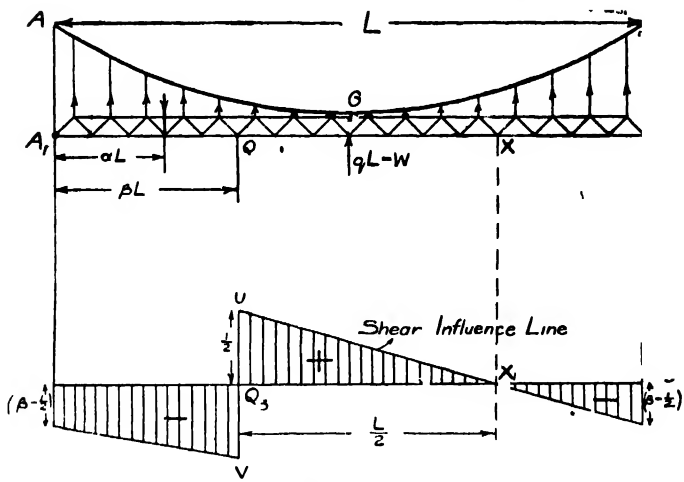


Fig. 41.

Influence Lines for Stiffened Suspension Bridges.

At the point C_1 where $\alpha = 1$, the ordinate will be equal to $\left(\beta - \frac{1}{2}\right)$ and at Q, where $\alpha = \beta$, it will be equal to $+\frac{1}{2}$.

The shear influence line is, therefore, as shown in Fig. 41.

Maximum Shear for Uniform Rolling Load.—It is clear that the maximum positive shear at Q for a uniform rolling load occurs when the load extends from Q to X.

$$\text{Then max. } S_Q = p \times \text{area of } \Delta\, U Q_3 X_1$$

$$= \frac{p \cdot L}{8}$$

This is independent of the position of Q and so is the same for all positions of Q.

The maximum negative shear at Q will occur when the portions $A_1 Q$ and $X C_1$ are loaded, and will have the same numerical value as the maximum positive shear; this will be clear from the influence line.

The curves therefore of maximum positive and negative shears will therefore be rectangles of height $\frac{pL}{8}$.

CHAPTER V.

INTERNAL WORK: DEFLECTIONS OF FRAMED STRUCTURES.

WHEN a structure is loaded a deflection results, and each of the loads moves a certain distance and does a certain amount of *external work;* each of the members of the structures becomes strained, and in becoming strained absorbs *internal work* or *resilience*. According to what is known as the *Principle of Work*, the external work done upon a structure must be equal to the internal work absorbed in straining it, and by application of this principle we are able to calculate the deflection of a structure.

Example of Simple Angle Bracket.—Take as an example the simple angle bracket shown in Fig. 42, the portion A C being regarded as rigid. Due to the load W there is a tensile stress in A B, and a compressive stress in B C, thus causing A B to stretch and B C to contract.

Calling A B the bar 1 and B C the bar 2, then if F_1, F_2 are the *forces* in the bars (commonly but not quite correctly called the stresses), A_1 A_2 are their areas; l_1 l_2 their lengths; x_1 x_2 their extensions; and E_1 and E_2 their Young's Moduli; we shall have

$$x_1 = \frac{F_1 \cdot l_1}{A_1 E_1} \qquad \text{(1)}$$

$$x_2 = \frac{- F_2 l_2}{A_2 E_2} \qquad \text{(2)}$$

Now the work done in straining will be equal to $\frac{1}{2} F_1 x_1 + \frac{1}{2} F_2 x_2$; it is $\frac{1}{2}$ because the strain and force increase gradually so that the stress-strain diagram is a triangle.

Therefore the work done

$$= \frac{1}{2} \frac{F_1^2 l_1}{A_1 E_1} + \frac{1}{2} \frac{F_2^2 l_2}{A_2 E_2}, \qquad \text{(3)}$$

the plus sign being used because both F_2 and x_2 are negative, or for an indefinite number of members we may write.

$$\text{Internal work} = \frac{1}{2}\Sigma\left(\frac{F^2 l}{AE}\right) \quad \text{.........................(4)}$$

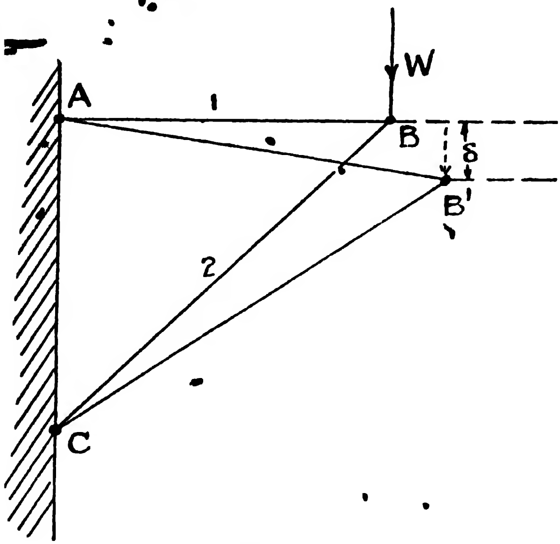


Fig. 42.

But external work $= \frac{1}{2} W \delta$ ($\frac{1}{2}$ because the load is considered to be applied gradually as deflection takes place).

$$\therefore \delta = \frac{F_1^2 l_1}{W A_1 E_1} + \frac{F_2^2 l^2}{W A_1 E_1}$$

Now F_1 and F_2 will be proportional to W, and if U_1 and U_2 are the forces in the bars caused by *unit* load, we have $U_1 = \frac{F_1}{W}$ and $U_2 = \frac{F_2}{W}$

$$\therefore \ \delta = W \left(\frac{U_1^2 l_1}{A_1 E_1} + \frac{U_2^2 l_2}{A_2 E_2} \right) \quad \ldots\ldots\ldots\ldots (5)$$

or more generally

$$\delta = W \Sigma \frac{U^2 l}{A E} \quad \ldots\ldots\ldots\ldots (6)$$

NUMERICAL EXAMPLE—*Take* AB = 5 *ft. and* $\angle$ACB = 45°; *if* W = 7 *tons and the areas of* AB *and* BC *are* 1 *sq. in. and* $3\frac{1}{2}$ *inches respectively, find the vertical deflection at the point* B, *taking* E = 12,500 *tons per sq. in.*

In this case $F_1 = W$ and $F_2 = W\sqrt{2}$

$$\therefore \ U = 1 \text{ and } U_2 = \sqrt{2}$$

$$\therefore \ \delta = \frac{7}{12{,}500} \left(\frac{1 \times 5 \times 12}{1} + \frac{2 \times 5 \times 12 \times \sqrt{2}}{3 \cdot 5} \right)$$

$$= \frac{7 \times 60}{12{,}500} \left(1 + \frac{2\sqrt{2}}{3 \cdot 5} \right)$$

$$= \frac{7 \times 60 \times 1 \cdot 81}{12{,}500}$$

$$= \cdot 061 \text{ inches.}$$

More general Case.—In the example that we have just considered there was only one load and we required the deflection only in the direction of that load. We will now consider the more general case.

Let Fig. 43 represent any structure the deflection of which is required in any direction *a*.

Take any bar 1 and imagine for the time being that all the other bars are rigid.

Then if a unit load is applied at x in the direction *a* and causes a deflection δ_1, then $\frac{1}{2} \times 1 \times \delta_1$ = external work done by unit force

$$= \frac{1}{2} U_1 x_1$$

where U_1 = load in bar 1 due to unit load at x

x_1 = extension " 1 " " "

$$i.e. \quad \delta_1 = U_1 x_1 \quad \ldots\ldots\ldots\ldots (7)$$

If therefore instead of putting the unit load at x we had stretched the bar 1 by an amount x_1 we should have obtained the same deflection δ_1 at x.

From this we get the following rule.

The deflection in any direction of any point in a framed structure due to an extension x in any one bar is equal to the load in that bar caused by unit load at the given point in the given direction multiplied by the extension x.

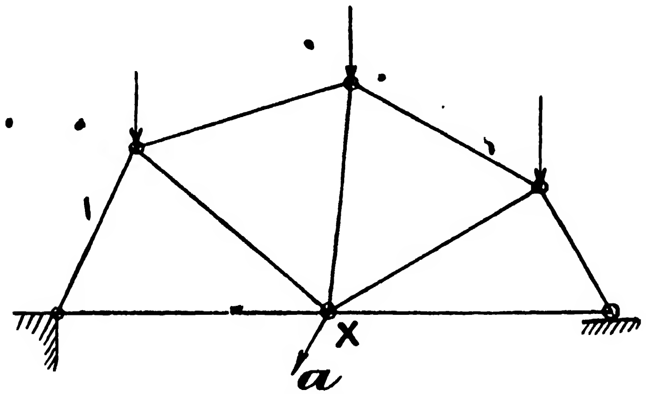


Fig. 43.

What is true for one bar is true for all, so that we may write

$$\delta = \Sigma \, U x \quad \text{.........................} (8)$$

Now x in an ordinary loaded structure will be equal to $\frac{F\,l}{A\,E}$, where F = load in any bar due to the loading and l, A, E are the length, area, and Young's modulus respectively.

Thus we get the formula in the form

$$\delta = \Sigma \left(\frac{U\,F\,l}{A\,E} \right) \quad \text{..............................} (8)$$

Our procedure to get the deflection in any direction at any point in a loaded framed structure is therefore as follows :

First find the extension in each bar due to the loading on the

structure; then take a unit load acting in the given direction and treat it as the only force on the structure and find the load due to it in each bar; multiply each extension by the load produced by the unit load and add the results together.

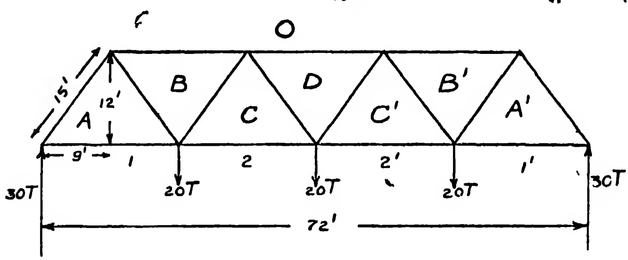
NUMERICAL EXAMPLE.—*Take the example of the previous paragraph and find the horizontal deflection.*

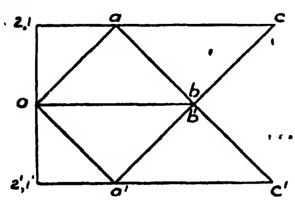
A unit horizontal force at B causes unit load in A B and no load in B C.

$$\therefore \quad U_1 = 1 ; U_2 = 0 .$$

$$x_1 = \frac{7 \times 12 \times 5}{1 \times 12{,}500}$$

$$\therefore \quad \delta = \frac{1 \times 7 \times 60}{12{,}500} = \cdot 034 \text{ inches.}$$





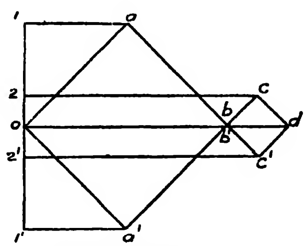


Fig. 44.—Deflection of Warren Girder.

Deflection of Warren Girder.—The calculation of deflections for a frame with a number of loads will be clear from the following calculation of the central deflection of a Warren girder of 72 ft. span loaded as shown in Fig. 44.

The loads (or stresses) F in the various bars are first found for the given loading either by the reciprocal figure method or by the method of moments.

The loads U in the various bars are then found for a unit load at the centre, the reciprocal figure in this case being drawn to a larger scale.

The results are then tabulated as follows :—

Member	Length l (ins.)	Area A (sq. ins.)	Load or Stress F (tons)	Unit Load or Stress U (tons)	$\frac{U \times F \times l}{A}$
O A, O A′	2 × 180	7·5	− 37·5	− ·625	1125
O B, O B′	2 × 216	9	− 45·0	− ·750	1620
O D	216	12	− 60·0	− 1·500	1620
1 A, 1′ A′	2 × 216	4·5	+ 22·5	+ ·375	810
2 C, 2′ C′	2 × 216	10·5	+ 52·5	+ 1·125	2430
A B, A′ B′	2 × 180	7·5	+ 37·5	+ ·625	1125
B C, B′ C′	2 × 180	2·5	− 12·5	− ·625	1125
C D, C′ D′	2 × 180	2·5	+ 12·5	+ ·625	1125

(+ indicates tension ; − compression.) Total $= \Sigma \frac{U F l}{A} = 10{,}980$

$$\therefore \text{ Central deflection } = \frac{10{,}980}{E} = \frac{10\,980}{12{,}500} = \cdot 88 \text{ ins. nearly.}$$

To save time and space in the above table, bars in which the stresses are the same from considerations of symmetry are tabulated together, the length of each being multiplied by two.

Deflection of Warren Girder by Formula.—If we assume certain working stresses for tension and compression (f_t and f_c respectively) we can deduce the deflection for any type of framed structure under uniform load. Take for instance the case of the Warren girder (Fig. 45).

Let d = length of each bay.
$2n$ = number of bays.
h = height of girder.

Then for tension members we have $\dfrac{F}{A} = f_t$

" " compression " " $\dfrac{F}{A} = f_c$

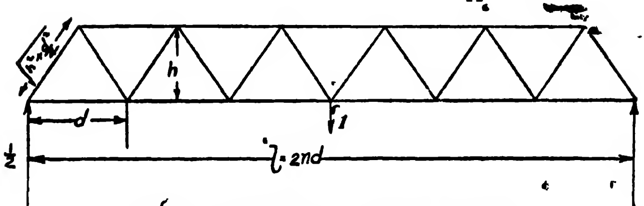


Fig 45.—Deflection of Warren Girder.

Consider first the lower Flange.

For the unit load at centre,

$$\text{U for first bay} = \frac{d}{4h} \qquad \text{[by method of moments]}$$

$$\text{U for second bay} = \frac{3d}{4h}$$

$$\text{U for third bay} = \frac{5d}{4h}$$

$$\text{U for } n^{\text{th}} \text{ bay} = \frac{(2n-1)d}{4h}$$

$$\therefore \quad \Sigma\frac{\text{U F } l}{\text{A}} \text{ up to centre} = \frac{d \cdot f_t \cdot d}{4h}\left\{1 + 3 + 5 \ldots (2n-1)\right\}$$

$$= \frac{n^2 d^2 f_t}{4h}$$

$$\therefore \Sigma\frac{\text{U F } l}{\text{A}} \text{ for whole span} = \frac{n^2 d^2 f_t}{2h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Upper Flange.

$$\text{U for first bar} = \frac{d}{2h}$$

$$\text{U for second bar} = \frac{2d}{2h}$$

$$\text{U for centre bar} = \frac{n\,d}{2\,h}$$

$$\therefore\ \Sigma\,\frac{\text{U F}\,l}{\text{A}}\ \text{up to centre} = \frac{f_c\,d^2}{2\,h}\left(1 + 2 \ldots n\right)$$

$$= \frac{f_c\,d^2\,n\,(n+1)}{2\,h\,.\,2}$$

For whole span we must add the above to that up to the $(n-1)$th bar

$$\therefore\ \Sigma\,\frac{\text{U F}\,l}{\text{A}}\ \text{for whole span} = \frac{f_c\,d^2}{2\,h}\left\{\frac{n\,(n+1)}{2} + \frac{n\,(n-1)}{2}\right\}$$

$$= \frac{f_c\,n^2\,d^2}{2\,h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Diagonal Compression Members.

$$\text{U for first bay} = \frac{\frac{1}{2} \times \sqrt{h^2 + \frac{d^2}{4}}}{h} = \frac{\sqrt{h^2 + \frac{d^2}{4}}}{2\,h}$$

This is the same for every bay.

$$\therefore\ \Sigma\,\frac{\text{U F}\,l}{\text{A}}\ \text{for whole span} = f_c\,.\,\sqrt{h^2 + \frac{d^2}{4}}\,.\,2\,n\,\frac{\sqrt{h^2 + \frac{d^2}{4}}}{2\,h}$$

$$= \frac{n\,f_c}{h}\left(h^2 + \frac{d^2}{4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Diagonal Tension Members.

U will be the same for these as for the compression members.

$$\therefore\ \Sigma\,\frac{\text{U F}\,l}{\text{A}}\ \text{for whole span} = \frac{n\,f_t}{h}\left(h^2 + \frac{d^2}{4}\right) \quad \ldots\ldots\ldots\ldots(4)$$

Adding up (1) to (4) and dividing by E we get

$$\delta = \frac{1}{\text{E}}\left\{\frac{n^2\,d^2}{2\,h}\left(f_c + f_t\right) + \frac{n}{h}\left(h^2 + \frac{d^2}{4}\right)\left(f_c + f_t\right)\right.$$

$$= \frac{f_c + f_t}{\text{E}}\left\{\frac{n^2\,d^2}{2\,h} + \frac{n}{h}\left(h^2 + \frac{d^2}{4}\right)\right\}$$

$$= \frac{f_c + f_t}{2\,\text{E}\,h}\left\{n^2\,d^2 + 2\,n\left(h^2 + \frac{d^2}{4}\right)\right\}$$

$$= \frac{f_c + f_t}{2\,E\,h}\left\{n\left(n + \frac{1}{2}\right)d^2 + 2\,n\,h^2\right\} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$= \frac{f_c + f_t}{2\,E\,h}\left\{\frac{l^2}{4} + \frac{l\,d}{4} + \frac{l\,h^2}{d}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5a)$$

We can apply this to the girder in Fig. 44, because it is clear from the table on page 93 that $f_c = f_t = 5$.

$$\text{Then } \delta = \frac{10 \times 144}{2 \times 12{,}500 \times 144}\left(36^2 + 18^2 + 4 \times 12^2\right)$$

$$= \frac{10 \times 2196}{2 \times 12{,}500}$$

$$= \cdot 88 \text{ in.}$$

DEFLECTION OF PRATT TRUSS BY FORMULA.—A similar treatment to the above applied to the Pratt Truss (see Fig. 45*a*) gives

$$\delta = \frac{f_c + f_t}{2\,E\,h}\left\{(n + 1)\,n\,d^2 + 2\,(n - 1)\,h^2\right\} \quad \ldots\ldots (6)$$

$$= \frac{f_c + f_t}{2\,E\,h}\left\{\frac{l^2}{4} + \frac{l\,d}{2} + \left(\frac{l}{d} - 2\right)h^2\right\} \quad \ldots\ldots\ldots (6a)$$

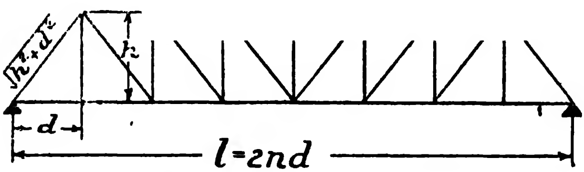


Fig. 45a.—Deflection of Pratt Truss.

COMPARISON WITH BEAM FORMULÆ.—It is of considerable interest to compare the deflections worked in the above method with those according to the ordinary beam deflection formulæ.

For a beam of uniform strength $\delta = \frac{M\,l^2}{8\,E\,I}$, and if y is the distance from the neutral axis to the point where the stress is f, we have $\frac{f}{y} = \frac{M}{I}$

$$\therefore \delta = \frac{f\,l^2}{8\,E\,y}$$

But if f_c and f_t are the extreme compression and tension stresses, we shall have $\frac{f_c + f_t}{h} = \frac{f}{y}$; this will be seen clearly if we consider the ordinary triangular diagram showing the distribution of stress in a beam;

$$\therefore \quad \delta = \frac{(f_c + f_t)\, l^2}{8\,E\,h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

If this be compared with the formula for the Warren girder, it will be noted that it is equal to the deflection contributed by the top and bottom flanges of the Warren girder.

To get a rough estimate of the error involved in applying this formula to the case of a Warren girder, take $h = \frac{l}{8}$ and $\frac{h}{d} = \frac{\sqrt{3}}{2}$, this corresponding to equilateral triangles in the truss.

$$\text{Then } \frac{l\,h^2}{d} + \frac{l\,d}{4} = l^2\left\{\frac{\sqrt{3}}{16} + \frac{1}{16\sqrt{3}}\right\} = \frac{l^2}{4\sqrt{3}}$$

$$\therefore \text{ equation } (5) = \frac{f_c + f_t}{2\,E\,h}\left(\frac{l^2}{4} + \frac{l^2}{4\sqrt{3}}\right)$$

Therefore the theoretical error of using equation (7) instead of equation (5) would be $\frac{l^2}{4\sqrt{3}}$ in $\frac{l^2}{4}\left(1 + \frac{1}{\sqrt{3}}\right)$, or roughly 36 %.

For a Pratt truss the error will come rather more than this; this analysis shows that the deflections in framed girders cannot be computed by the ordinary beam formulæ without introducing considerable error.

Experiments on American pin-connected truss bridges have shown that the actual deflections agree quite well with the formulæ; with riveted truss bridges that are usual in British practice the deflections will come a little less than the calculated deflections, and the calculated deflections for plate girders are always somewhat less than the actual deflections, so that in practice the difference between the deflections of plate girders and Warren girders will not be as great as indicated in the above treatment.

Height of Girder for Maximum Stiffness.—We can find the height of a Warren girder to give maximum stiffness by

differentiating equation (5) with respect to h and equating to zero. We then get :

$$\frac{d\,\delta}{d\,h} = \frac{(f_c + f_t)}{2\,\mathrm{E}}\left\{-\frac{l^2}{4\,h^2} + \frac{l}{d} - \frac{l\,d}{4\,h^2}\right\} = 0$$

$$\therefore \quad \frac{-\,l\,(d + l)}{4\,h^2} + \frac{l}{d} = 0$$

$$h^2 = \frac{d\,(d + l)}{4}$$

$$h = \frac{\sqrt{d\,(d + l)}}{2}$$

$$= \frac{d}{2}\sqrt{1 + \frac{l}{d}} \qquad \text{.................}(8)$$

A similar analysis for the Pratt truss gives

$$h = \frac{d}{2}\sqrt{2\,n\left(\frac{n + 1}{n - 1}\right)}$$

$$= \frac{d}{2}\sqrt{\frac{l}{d}\cdot\frac{\frac{l}{d} + 2}{\frac{l}{d} - 2}} \qquad \text{..........}(9)$$

The following tables give values of $\frac{h}{d}$ for various values of $\frac{l}{d}$ to give maximum stiffness or rigidity :—

WARREN GIRDER.

No. of Bays $= \frac{l}{d}$	4	6	8	10	12
$\frac{h}{d}$	1·12	1·32	1·50	1·66	1·80

PRATT TRUSS.

No. of Bays $= \frac{l}{d}$	4	6	8	10	12
$\frac{h}{d}$	1·73	1·73	1·82	1·94	2·05

It follows from the following reasoning that *when the stress per square inch in a girder is constant the girder of maximum stiffness is also that of minimum weight.*

The total internal work $= \Sigma \frac{F^2 l}{A E}$

But $\frac{F}{A} =$ stress per square inch $= f$ (assumed constant).

$$\therefore \text{ Total internal work } = \Sigma \frac{f^2 a l}{E}$$

$$= \frac{f^2 \Sigma a l}{E}$$

$$= \frac{f^2 \times \text{volume of girder}}{E}$$

Now, if the girder is of maximum stiffness, the deflections, and therefore total external work, will be a minimum; therefore the total internal work will be a minimum, and thus the volume and weight of the girder will be a minimum.

In using the above formulæ for deflections and the values of $\frac{l}{d}$ for maximum economy, it must be remembered that assumptions have been made that the stress is constant throughout the truss; this is very seldom attained in practice, principally because the compression members have to have their buckling factors (length in terms of least radius of gyration for pin-jointed ends) taken into consideration in the determination of the stresses. In practice the height is usually less than that given for maximum economy; this is justified when the question of the length of the compression members is taken into account.

GRAPHIC DETERMINATION OF DEFLECTIONS.

The deflections of framed structures may be found by the following graphical constructions, which are analogous to the velocity diagrams employed in investigating the kinematics of mechanisms; they are sometimes referred to as Williott diagrams.

We will return to the simple case of the angle bracket. The bar 1, Fig. 46, stretches by an amount B 1′, and the bar 2 contracts

by an amount B 2′; the bar 1 can turn about A, and the bar 2 can turn about C, so that by drawing arcs from 1′ and 2′ with centres at A and C respectively, we get at their intersection B′ the position which B will take up under the load W. Now, the lengths B 1′ and B 2′ are in reality extremely small, so that the arcs 1′ B′ and 2′ B′ can be replaced by straight lines at right angles to B 1′, and B 2′ respectively; we thus get the following construction:

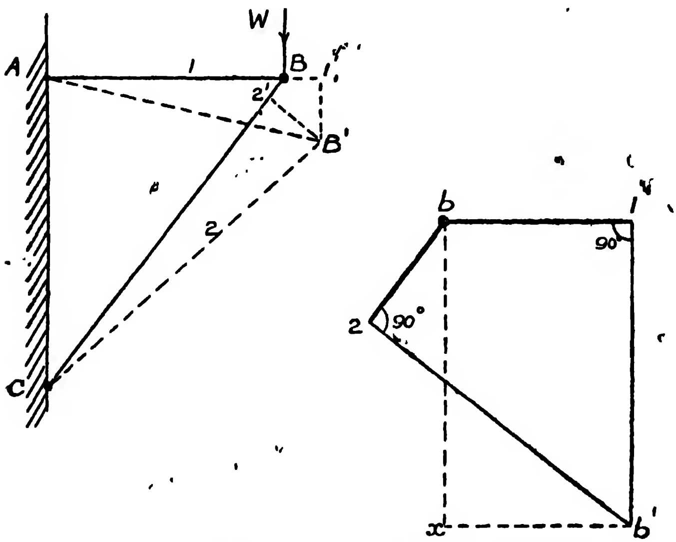


Fig. 46.—Graphical Determination of Deflections.

Set out to a much enlarged scale lengths *b* 1, *b* 2 parellel to their respective bars to represent the strains in the bars 1 and 2; then draw 1 *b*′ and 2 *b*′ perpendicular to *b* 1, *b* 2. Their intersection *b*′ gives the displaced position of *b*, and if *b x* is vertical and *b*′ *x* is horizontal, *b x* will be the vertical deflection of the point B and *x b*′ will be horizontal deflection.

Application to Warren Girder.—Now consider the application of the method to the Warren girder which we considered on p. 92. We have used different notation for this on

Fig. 47 because it is rather more convenient to number the bars in the present construction.

For the graphical construction we require to know the strains; we will therefore tabulate as follows:—

Member.	Length l (inches)	Area A (sq. ins.)	Load or Stress F (in tons)	Strain $= \frac{F\,l}{A\,E}$ (inches)
1, 1′	180	7·5	− 37·5	− ·072
4, 4′	216	9	− 45·0	− ·086
8,	216	12	− 60·0	− ·086
2, 2′	216	4·5	+ 22·5	+ ·086
6, 6′	216	10·5	+ 52·5	+ ·086
3, 3′	180	7·5	+ 37·5	+ ·072
5, 5′	180	2·5	− 12·5	− ·072
7, 7′	180	2·5	+ 12·5	+ ·072

As the loading is quite symmetrical we will work from the centre, and treating the point K as fixed we will find the relative upward deflections of the other points, working towards F.

On the displacement diagram set out k 8 to represent *half* the strain in the bar 8, half being taken because we are considering only that part of the girder to the left of the dotted line; also set out k 7 to represent the strain in K J and draw perpendiculars at 7 and 8, their intersection giving the point j. (So far the case is exactly similar to the angle bracket.) We next require to find the displaced position of the point H; therefore set out k 6 to represent the strain in 6 and j 5 that in 5 and draw perpendiculars to get the intersection h; h 3 and j 4 are then drawn to represent the strains in 3 and 4 respectively and the point g thus obtained, the final point f being found in a similar manner.

To make sure in which direction to draw the strain for any bar, say 7, starting from one end K, consider in which direction

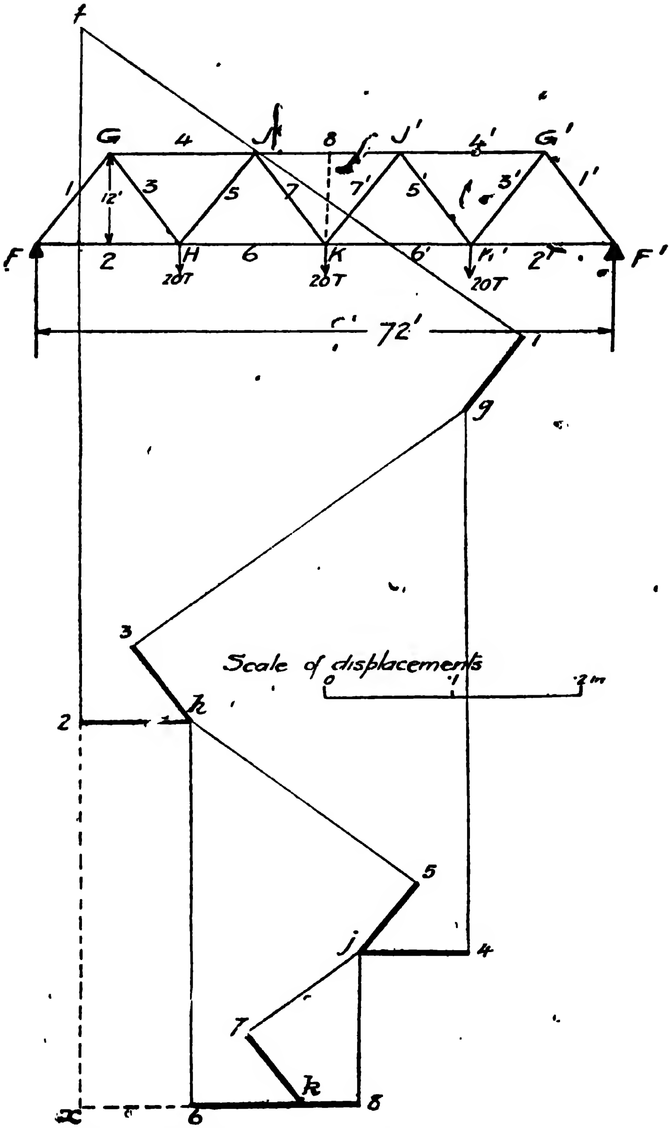


Fig. 47.—Displacement Diagram for Warren Girder with Load Symmetrical.

the strain goes from the other end J; if it is a tension strain, draw towards J; if compression strain, draw away from it.

Then $f \cdot x$ will give the displacement of the point F relatively to k, *i.e.*, equal to the deflection of K, and will be found to scale off equal to ·88 inches. If we completed the strain diagram for the other side we should get a reflection of the displacement diagram, the point f' coming in a horizontal line with the point f.

The displacement diagrams possess the advantage over the methods of calculation previously discussed that they give the displacement of every point in the structure and do not give the deflection at one point only.

Loading not Symmetrical.—If the loading is not symmetrical we have to work from both sides, and some adjustment has to be made at the completion of the construction to make the deflections equal when worked from each end. Consider for instance the same truss as before, but a load of 60 tons at the first node instead of 20 tons at each.

The stresses can be obtained from the reciprocal figure shown in Fig. 48 or can be calculated easily by the method of moments.

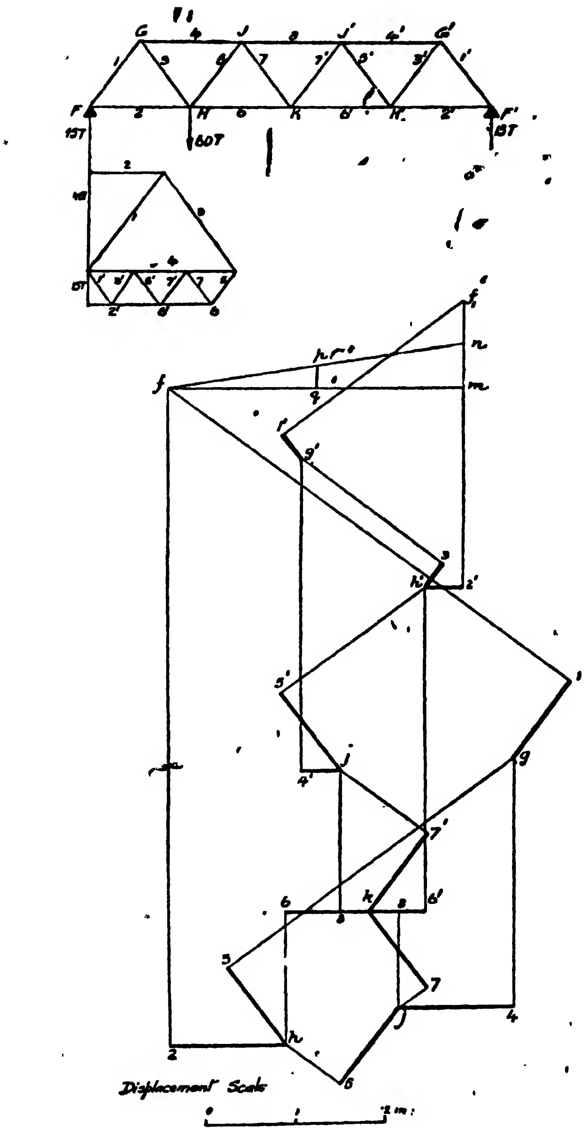


Fig. 48.—Displacement Diagram for Warren Girder with Load not Symmetrical.

We can then tabulate the strains as follows :—

Member.	Length l (ins.)	Area A (sq. ins.)	Load or Stress F (in tons)	Strain $= \frac{F\,l}{A\,E}$ (inches)	$\frac{U\,F\,l}{A\,E}$
1	180	7·5	− 56·25	− ·108	+ ·0675
2	216	4·5	+ 33·75	+ ·130	+ ·0486
3	180	7·5	+ 56·25	+ ·108	+ ·0675
4	216	9	− 67·5	− ·130	+ ·0972
5	180	2·5	+ 18·75	+ ·108	− ·0675
6	216	10·5	+ 56·25	+ ·093	+ ·1042
7	180	2·5	− 18·75	− ·108	− ·0675
8	216	12	− 45·00	− ·065	+ ·[illegible]75
7′	180	2·5	+ 18·75	+ ·108	+ ·0675
6′	216	10·5	+ 33·75	+ ·062	+ ·0708
5′	180	2·5	− 18·75	− ·108	+ ·0675
4′	216	9	− 22·50	− ·043	+ ·0326
3′	180	7·5	+ 18·75	+ ·036	+ ·0215
2′	216	4·5	+ 11·25	+ ·043	+ ·0162
1′	180	7·5	− 18·75	− ·036	+ ·0215

The last column is added in order to check the central deflection by calculation, the values of U being obtained from the table on p. 93. In connection with this column there will be noted a new point which has not arisen in the examples considered up to the present; the values of F and U for the bars 5 and 7 come of opposite sign, so that the value of $\frac{U\,F\,l}{A\,E}$ comes negative for these bars.

To draw the displacement diagram we start as before at the point *k* and working towards the left we come to the point *f*; then working towards the other side we reach the point f_1.

As a rule with non-symmetrical load the points f, f_1 will not come on the same horizontal line as of course they should be if F and F′ are at the same level, $f\,m$ representing the difference in level; if we bisect $f\,m$ at n and join $f\,n$, then $n\,m$ represents the amount by which we have to correct the deflection at the point K. To get the corrections for other points we will note that the correction consists in swinging the whole girder about the point K by the amount $n\,m$ upwards at F and $n\,f_1$ downwards at F′; the whole girder can then be drawn on this inclined base, the displacement scale being preferably reduced if $n\,m$ is more than about $\frac{1}{20}$ span. As a rule, however, only the deflections of points on the lower flange are required; in this case the following rule will give the corrections. Divide $f\,m$ into as many equal parts as there are bays between K and each end, and erect perpendiculars such as $p\,q$ from each to $f\,n$; then these perpendiculars measure the amounts that must be added or subtracted to the deformations of flange points measured from the point n; if the flange points are on the same side of K as F, the distances $p\,q$ are subtracted from the measurement to the point n, but if on the same side as F_1, the distances are added to the measurement to the point.

The central deflection in the above example will be equal to the vertical distance between the points k and n; this measures ·65 in. and agrees with the result obtained by adding up the column of $\frac{U\,F\,c}{A\,E}$ in the table.

The maximum deflection occurs at the point H and will be equal to the vertical distance between h and n *minus* the distance $p\,q$; this comes equal to ·78 in.

We could correct the displacement diagram by redrawing it with the point k to the right by an amount equal to $\frac{n\,m \times 2\,h}{l}$ where h is the height of the girder and l is the span; the diagram so redrawn would have the points f and f_1 on the same horizontal line.

Displacement Diagram for Roof Truss.—We will next consider the application of the displacement diagram to find the horizontal movement of a roof truss provided with a roller bearing; in this problem we are concerned principally with the movement

due to the wind, because once the roof has been erected there will be no movement due to the dead load.

Take, for example, the roof truss of 40 ft. span shown in Fig. 49. We may note in passing that from a practical point of view this truss is not very economical; the struts have been kept vertical for improved appearance, but this has necessitated different lengths for several members such as 10, 11 and 7, 12.

The principals are 10 feet apart, and for a 'vertical' wind pressure of 30 lbs. per sq. ft. the total wind force comes equal to 1·86 tons, which is divided up as shown in the figure. Considering the case where the wind blows towards the fixed end, we get in the ordinary manner the stress diagram shown.

We can then tabulate as follows:—

Member.	Length l (inches).	Area A (sq. inches).	Load or Stress F (tons).	Strain $= \frac{F\,l}{E\,A}$ (inches).
1	92·3	2·50	− 2·48	− ·0073
2	92·3	2·50	− 1·87	− ·0055
3	92·3	2·50	− 2·23	− ·0066
4	92·3	2·50	− 1·45	− ·0043
5	92·3	2·50	− 1·45	− ·0043
6	92·3	2·50	− 1·45	− ·0043
7	161·3	1·31	+ 2·95	+ ·0291
8	158·4	1·31	+ 1·12	+ ·0108
9	161·3	1·31	+ 1·27	+ ·0125
10	83·0	1·44	− 0·85	− ·0039
11	67·7	1·44	− 0·72	− ·0027
12	138·7	1·31	+ 1·68	+ ·0142
13	138·7	1·31	+ 0·23	+ ·0019
14	67·7	1·44	0·00	—
15	83·0	1·44	0·00	—

To draw the displacement diagram we start with the bar 8 and make $e\,e'$ = ·0108 on a convenient scale; we next fix the

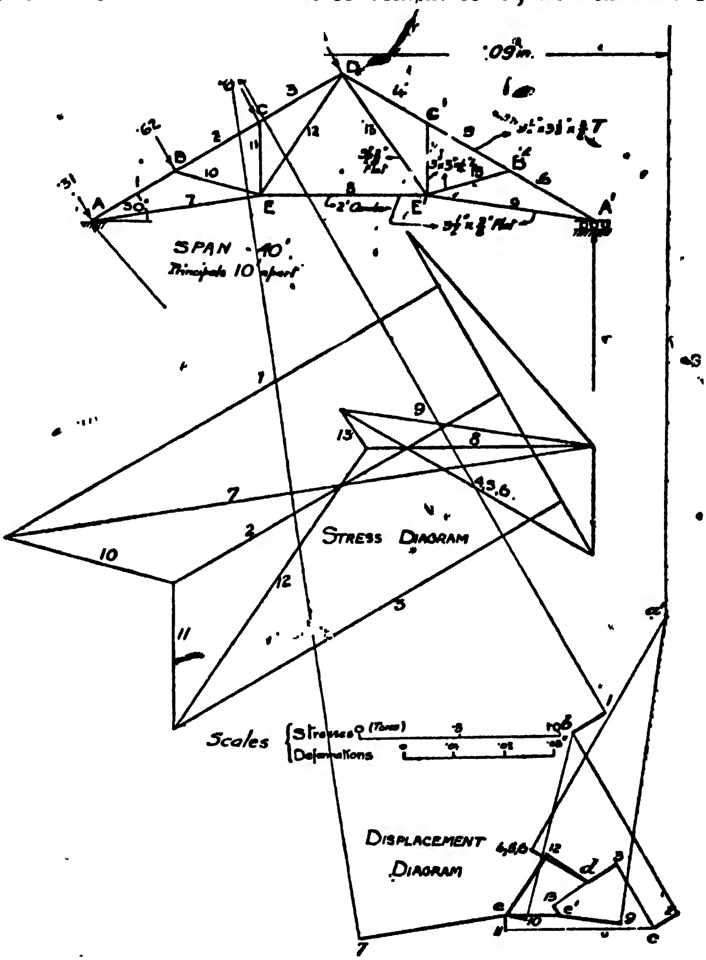


Fig. 49.—Displacement Diagram for a Roof Truss.

point d and working towards the left in the same manner as in previous examples the point a is reached without much difficulty;

working towards the right we note that there is no strain in 14 and 15, so we set out a length from d equal to the combined strains of 4, 5, 6 and reach without difficulty the point a'. Then the horizontal distance between the points a, a', measured on the displacement scale, gives us the amount of horizontal movement of the point A′ due to the wind blowing on the fixed end; this comes to about ·09 in.; a similar treatment will give the movement for the wind blowing in the other direction.

It will be noted that the movement is very small, so that a very simple form of expansion bearing would suffice to take up the movement.

Temperature Deflections.—If β is the co-efficient of linear expansion, and l is the length of any member, and t is change in temperature, the change in the length of the member due to temperature $= x = \beta\, t\, l$.

We may therefore apply the methods previously considered to find the deflections due to these strains; if, for instance, U is the stress in the given bar due to a unit load in a given direction at a certain point, then the deflection in that direction at that point will be equal to

$$\delta = \Sigma\, \mathrm{U}\, \beta\, t\, l$$

The displacement diagram could be used conveniently to determine such deflections due to temperature changes.

Application to Continuous Framed Girders.—The reactions in continuous framed girders of two spans can be found accurately as follows by means of the foregoing methods of obtaining deflections: First assume the centre support to be removed and find the resulting deflection δ at that point. Next take a unit upward load at the point as the only load on the frame and calculate the resulting upward deflection δ_1.

The central reaction required $= \dfrac{\delta}{\delta_1}$. This is equivalent to the method for solid continuous beams of two equal spans (A, p. 245).

CHAPTER V.

STRESSES IN REDUNDANT FRAMES.

THE stresses in redundant frames cannot be determined by the application of the ordinary methods of graphical or analytical statics, such frames forming with rigid arches, continuous beams, and some other structures a class which is often referred to as '*Statically Indeterminate Structures.*' In practice it is not usual to apply the more rigorous methods such as are outlined below, principally because the dimensions of the various members of the frame have to be known before the stresses in the members can be determined: the usual procedure is to work by the method of superposition, by which the redundant frame is considered as divided up into a number of superposed firm frames and the load divided between them, the stresses in common members being added together; the results obtained by this method are not usually very erroneous. In some other cases the diagonals are treated as semi-members which can resist only one kind of stress, and one or other of the diagonals is considered as coming into action according to the position of the load.

British engineers at one time were strongly in favour of redundant frames in bridge design on account of their additional rigidity and alleged additional safety, but recent practice tends more in favour of the firm or 'single-intersection' trusses.

Frames with a Single Redundant Member.—Take, for example, the simple frame shown in Fig. 50, and take one of the diagonals, say 6, as the redundant member. Then if F is the load or stress in any member, we may regard F as made up of two parts:

(1) A stress F′ due to the loads W_B and W_C with the bar 6 removed.

(2) A stress F″ due to a force applied at the point B, equal to the stress F_6 in the redundant bar 6 and in the direction of this bar.

Let U be the stress in any bar due to a unit force at B in the direction of the bar 6; then we may we may write

$$F = F' + F''$$
$$= F' + U_6 F_6 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

F' can, of course, be found easily by the ordinary methods.

Suppose that the displacement of the point B is equal to δ' in

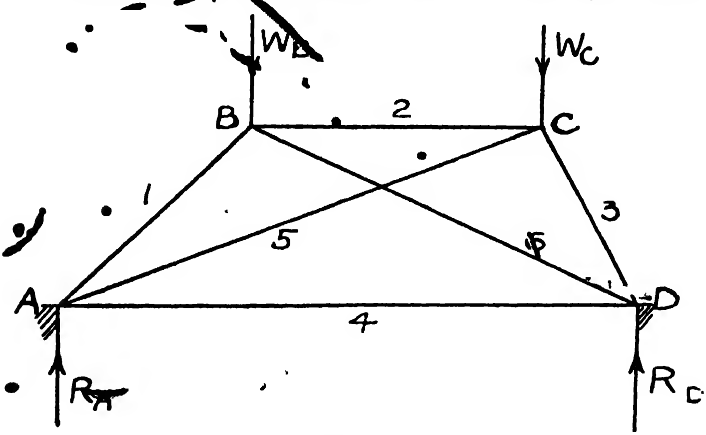


Fig. 50.—Redundant Frames.

the direction of the bar 6 when it is considered removed, then by the results of the previous chapter we have

$$\delta' = \sum_1^5 \frac{F' U l}{E A} \quad (1a)$$

Due to the force F_6 we shall have a movement equal to

$$\delta_1 = \sum_1^5 \frac{F_6 U^2 l}{E A} = F_6 \sum_1^5 \frac{U^2 l}{E A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

This is because the stress F in any given member due to a load F_6 will clearly be equal to F_6 U.

∴ Total displacement $= \delta = \delta' + \delta_1$

$$= \sum_1^5 \frac{F' U l}{E A} + F_6 \sum_1^5 \frac{U^2 l}{E A}$$

but clearly δ = strain in bar 6 $= \dfrac{-F_6 l_6}{E_6 A_6}$

This is *minus* because we have considered the force F_6 as inwards, so that the strain due to it will be a compressive or negative one.

$$\therefore\ F_6\left\{\frac{l_6}{E_6 A_6}+\sum_1^5\frac{U^2 l}{E A}\right\}=\sum_1^5\frac{F' U l}{E A}.$$

$$\therefore\ F_6=-\frac{\sum_1^5\left(\frac{F' U l}{E A}\right)}{\frac{l}{E_6 A_6}+\sum_1^5\frac{U^2 l}{E A}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

We can simplify this by noting that for the bar 6, $U = 1$, so that for this bar $\frac{U^2 l}{E A}=\frac{l_6}{E_6 A_6}$

We can therefore write

$$\frac{l_6}{E_6 A_6}+\sum_1^5\frac{U^2 l}{E A}=\sum_1^6\frac{U^2 l}{E A}$$

This gives us

$$F_6=-\frac{\sum_1^5\frac{F' U l}{E A}}{\sum_1^6\frac{U^2 l}{E A}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

If, as is common, E is the same for the various members of the frame, we may write

$$F_6=-\frac{\sum_1^5\frac{F' U l}{A}}{\sum_1^6\frac{U^2 l}{A}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4a)$$

Expressing these formulæ in more general terms, we have

$$F_{n+1}=-\frac{\sum_1^n\frac{F' U l}{E A}}{\sum_1^{n+1}\frac{U^2 l}{E A}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4b)$$

NUMERICAL EXAMPLE.—*Take for example the case shown in Fig.* 51 *and take the areas of the bars* 1, 2, 3 *as* 5 *square inches, and bar* 4 *and the diagonals as* 2 *square inches, and the lengths of* 1, 3 *as* 10 *feet, and* B D *and* A C *as* 16 *feet each.* In this figure (1) is the ordinary reciprocal figure for the loading, and (2) is that for unit load at B, in the direction of bar 6 (B D).

From the reciprocal figures shown, we are able to tabulate approximate results as follows :—

Member	l (inches)	A (sq. ins.)	F′ (tons)	U (tons)	$\frac{F' U l}{A}$	$\frac{U^2 l}{A}$
1	120	5	– 5·06	– ·625	+ 75·9	9·4
2	93·6	5	– 3·08	– 1·25	+ 72·1	29·2
3	120	5	– 5·93	– ·625	+ 88·9	9·4
4	240	2	+ 3·62	– ·49	– 212·9	28·8
5 (A C)	192	2	– 0·64	+ 1·00	– 61·4	96·0
6 (B D)	192	2	—	+ 1·00	—	96·0
			Totals ...	...	– 37·4	268·8

$$\therefore F_a \quad \frac{-37·4}{268·8} = + ·139 \text{ tons.}$$

We can now calculate the stress in the various members by means of equation (1), and can tabulate as follows :—

Member	Stress by correct formula (tons)	Stress by superposition (tons)	Numerical error in superposition (tons)
1	– 5·15	– 5·19	+ ·04
2	– 3·25	– 3·46	+ ·21
3	– 6·02	– 6·12	+ ·10
4	+ 3·55	+ 3·47	– ·08
5	– ·50	– ·32	– ·18
6	+ ·14	+ ·32	+ ·18

For the stresses by superposition we use reciprocal diagram 1, using the dotted lines for the diagonals reversed and taking the mean values.

It is clear from the above that in the present case, which may be

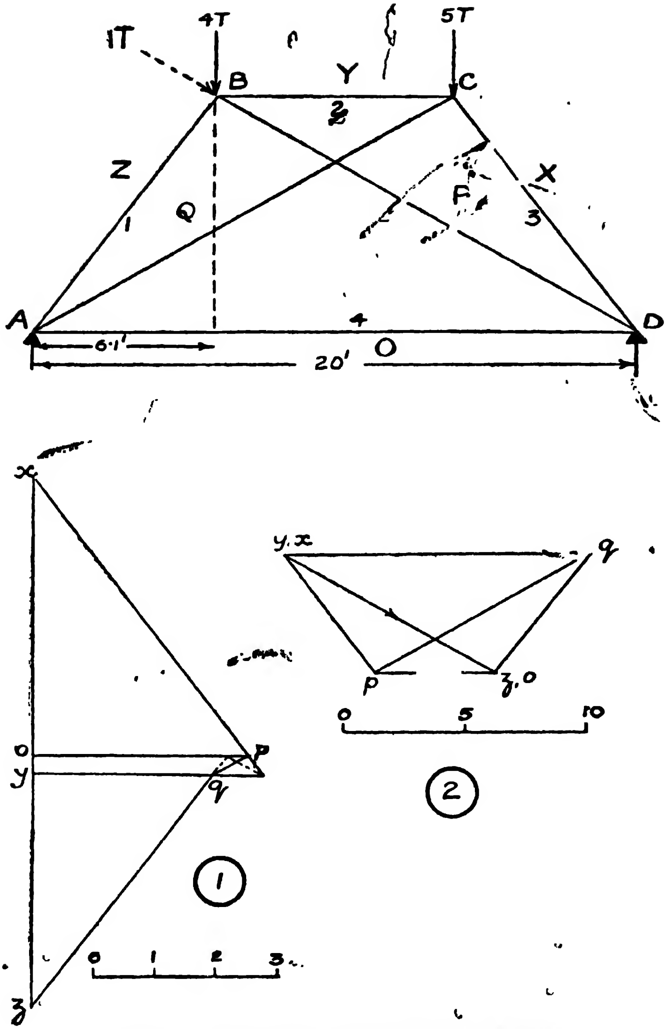


Fig. 51.—Example of Redundant Frame.

taken as typical, the error involved in the superposition method is almost too small to worry about when there are errors such as those

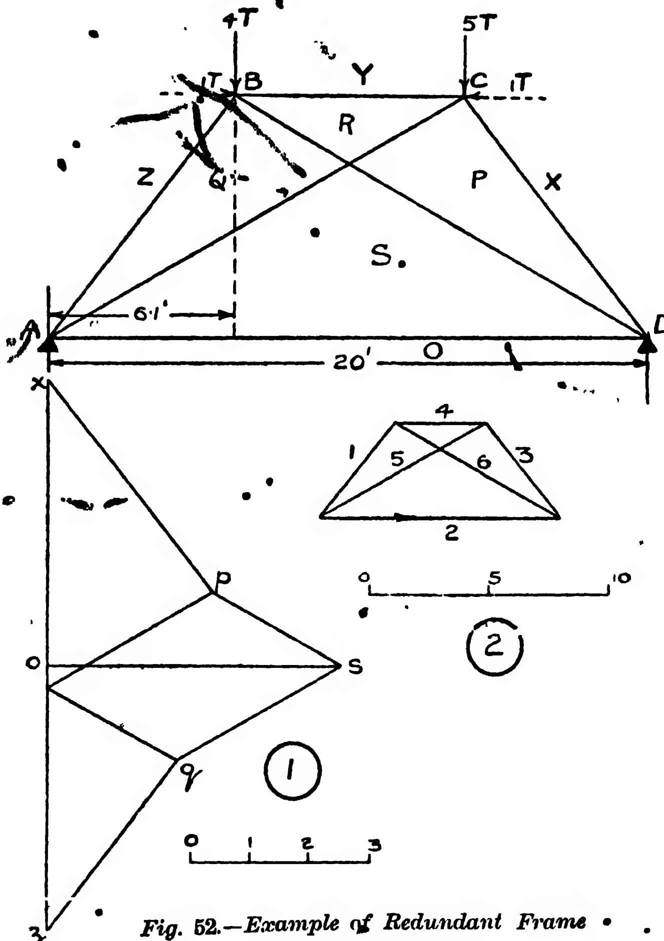


Fig. 52.—Example of Redundant Frame (alternative solution).

due to stiffness of joints and continuity of flanges nearly always existing in most calculations for framed structures.

Alternative Solution.—As an interesting check upon the above

working, we will now recalculate the stresses on the assumption that the member 1 is the redundant one.

The reciprocal diagrams for load and unit stress are as shown in Fig. 52, and the stresses can be tabulated as follows :—

Member	l (inches)	A (sq. ins.)	F′ (tons)	U (tons)	$\frac{F' U l}{A}$	$\frac{U^2 l}{A}$
1	120	5	− 3·51	+ ·5	− 42·1	6·0
2	93·6	5	—	1	—	18·7
3	120	5	− 4·39	+ ·5	− 52·7	6·0
4	240	2	+ 4·90	+ ·39	+ 229·3	18·5
5	192	2	− 3·08	− ·80	+ 236·5	61·4
6	192	2	− 2·47	− ·80	+ 189·7	61·4
			Totals		560·7	172

$$F_2 = -\frac{560 \cdot 7}{172} = -3 \cdot 26$$

This agrees quite well with the previous working, and the stresses in the other bars will be found to be in good agreement.

***Frames with several Redundant Members.**—If there are a number of redundant members, we can continue the same principle as for one redundant bar.

Suppose that the number of members for a firm frame is n, and that redundant members are indicated by $(n + 1)$, $(n + 2)$, &c.

Then if U_1, U_2, &c., represent the stresses in any given bar due to unit loads in the direction of the first, second, &c., redundant bars at the nodes where such redundant bars come, we shall have

$$F = F' + U_1 F_{n+1} + U_2 F_{n+2} \text{ \&c.} \quad \ldots\ldots\ldots\ldots (5)$$

where F′ is the stress in the given bar when all the redundant members are removed and F is the stress required.

In finding the stresses U_1 we assume the other redundant bars removed and so on.

Considering the first redundant bar we shall have as before a deflection in the direction of this bar equal to

$$\delta' = \sum_{1}^{n} \frac{F' U_1 l}{E A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Due to the force F_{n+1} we shall have a further deflection of

$$\delta_1 = F_{n+1} \sum_{1}^{n} \frac{U_1^2 l}{E A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Due to the force F_{n+2} there will be a further deflection of δ_2 equal to

$$\delta_2 = F_{n+2} \sum_{1}^{n} \frac{U_1 U_2 l}{E A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This is because the force in any given bar is $F_{n+2}\, U_2$ and this will correspond to the term F in the general formula for deflection $\delta = \dfrac{\Sigma\, F\, U\, l}{E A}$.

$\therefore$ we have total deflection in direction of first redundant bar where there are two redundant bars

$$\delta = \sum_{1}^{n} \frac{F' U_1 l}{E A} + F_{n+1} \sum_{1}^{n} \frac{U_1^2 l}{E A} + F_{n+2} \sum_{1}^{n} \frac{U_1 U_2 l}{E A}$$

but as before $\qquad \delta = \dfrac{-\,F_{n+1}\, l_{n+1}}{E_{n+1}\, A_{n+1}}$

This gives us

$$F_{n+1} \sum_{1}^{n+1} \frac{U_1^2 l}{E A} + F_{n+2} \sum_{1}^{n} \frac{U_1 U_2 l}{E A} = -\sum_{1}^{n} \frac{F\, U_1 l}{E A} \quad \ldots\ldots (9)$$

A similar consideration of the second redundant bar will give us the equation

$$F_{n+2} \sum_{1}^{n+1} \frac{U_2^2 l}{E A} + F_{n+1} \sum_{1}^{n} \frac{U_1 U_2 l}{E A} = -\sum_{1}^{n} \frac{F^1 U_2 l}{E A} \ldots\ldots (10)$$

In any given case the above summations can be made and their values placed in the above equations, thus giving simultaneous equations for F_{n+1} and F_{n+2}.

If there are more than two redundant bars, we get similar simultaneous equations, their number being equal to the number of redundant members.

*** Stresses due to Errors in Length.**—In a redundant frame there will be internal stresses induced in the members if one of the members is not of exactly the correct length.

In such a case we can calculate the internal stresses in the following manner.

Let x be the amount by which a given member falls short of its requisite length (if it is too long, the excess will be considered negative), then if as before U is the stress in any given member due to a unit force in the direction of the member of incorrect length (tending to reduce the length), then the deflection of the given point due to this load will be given by $\sum_1^n \frac{U^2 l}{EA}$, when n is the number of bars in addition to the redundant one.

If F is the stress in the member caused by straining it into place, we shall have $F \sum_1^n \frac{U^2 l}{EA} + \frac{F l}{EA} = x$

This is because the bar will stretch by an amount $\frac{F l}{EA}$, and this stretch added to the inward deflection of the joint must be equal to x

$$\therefore F = \frac{x}{\frac{l}{EA} + \sum_1^n \frac{U^2 l}{EA}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

For the redundant bar U = 1 so we may write

$$F = \frac{x}{\sum_1^{n+1} \frac{U^2 l}{EA}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

*** The Principle of Least Work.**—Some treatments of the stresses in redundant frames are based upon the above principle. According to this principle, the stresses in a redundant frame are such that the total internal work or resilience is a minimum. We will illustrate this by the case of the frame with one redundant bar.

From equation (1) p. 111.

$$\sum_1^{n-1} \frac{F U l}{EA} = \sum_1^{n+1} \frac{F' U l}{EA} + F_{n+1} \sum_1^{n+1} \frac{U^2 l}{EA}$$

Now for bar $(n + 1)$, $F' = 0$

$$\therefore \sum_1^{n+1} \frac{F\,U\,l}{E\,A} = \sum_1^{n} \frac{F'\,U\,l}{E\,A} + F_{n+1} \sum_1^{n+1} \frac{U^2\,l}{E\,A}$$

but by equation $(4b)$

$$F_{n+1} \sum_1^{n+1} \frac{U^2\,l}{E\,A} + \sum_1^{n} \frac{F'\,U\,l}{E\,A} = 0$$

$$\therefore \sum_1^{n+1} \frac{F\,U\,l}{E\,A} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

A corresponding equation will hold for any number of redundant bars.

Now if P is the resilience or internal work in a bar

$$P = \frac{F^2\,l}{2\,E\,A}$$

If W is any load applied at a point

$$\frac{d\,P}{d\,W} = \frac{d\,P}{d\,F} \cdot \frac{d\,F}{d\,W} = \frac{2\,F\,l}{2\,E\,A} \cdot \frac{d\,F}{d\,W}$$

$$= \frac{F\,l}{E\,A} \cdot U$$

because U is the increment of stress for unit increment of load

$$\therefore \frac{F\,U\,l}{E\,A} = \frac{d\,P}{d\,W}$$

$\therefore$ by equation (13) $\dfrac{d\,P}{d\,W} = 0$, or the total resilience or internal work is a minimum.

CHAPTER VII.

STRESSES IN RIGID OR ELA[illegible] ARCHES.

By rigid or elastic arches are meant those in which the reactions cannot be obtained by purely statical considerations, the most common cases being those which are two-pinned (*i.e.*, having two hinges usually at each end) and those whose ends are fixed without hinges.

TWO-PINNED ARCH RIBS.

It can be shown that if a trial value H_0 of the horizontal thrust be taken, and the line of pressure A Q C, Fig. 53, be drawn for this trial value, and the arch rib A G C be divided up into a convenient number of equal parts, and mid-ordinates y, z be drawn to the arch and line of pressure respectively, then the real horizontal thrust H is given approximately by

$$H = \frac{H_0 \Sigma y z}{\Sigma y^2} \quad \ldots\ldots \quad (1)$$

$$= \frac{\text{'arch-load sum'}}{\text{'arch-square sum'}} \quad H_0$$

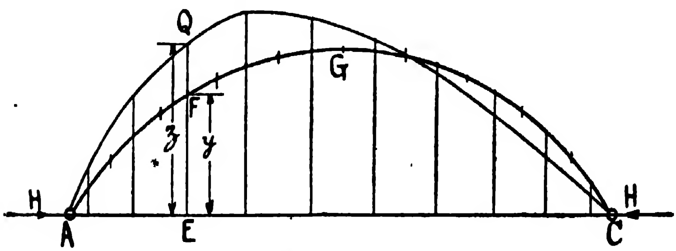


Fig. 53.—Two-pinned Arch Ribs.

This is true for flat arches only, and for other cases must be divided by $\left(1 + \frac{15}{8}\frac{k^2}{r^2}\right)$ where k is the radius of gyration of the rib and r is the rise.

This may be proved at considerable length without use of the methods of the calculus, and is given in A, p. 377 onwards, with the substitution of Y for *y* and *m* for *z*.

Formula obtained from Internal Work.—The following proof of the approximate formula involving the notation of the calculus will be found shorter and is similar to that employed in investigating the deflections of curved beams.

The work done by a couple in moving through an angle is equal to the product of the moment of the couple into the angle turned through. Therefore, if a short portion ds of a curved beam subjected to a bending moment M is caused to change its slope by an amount di, the work done in bending is $\frac{M\,di}{2}$, because M increases gradually from 0 to M.

$\therefore$ Total work done in bending

$$= W = \int \frac{M\,di}{2}$$

but $\frac{di}{ds} = \left(\frac{1}{R} - \frac{1}{R_0}\right)$ by the properties of the circle, where R, R_0 are the final and original radii of curvature of the element and $\left(\frac{1}{R} - \frac{1}{R_0}\right) = \frac{M}{EI}$ by the theory of bending (if the original curvature is not large as in hooks and rings).

$$W = \int \frac{M}{2} \cdot \frac{M\,ds}{EI} = \int \frac{M^2}{2\,EI} \cdot ds$$

$$\text{or } dW = \frac{M^2}{2\,EI}\,ds \qquad \text{.........................} \qquad (2)$$

Now let P be the imaginary load applied at any point, then

$$\frac{dW}{dP} = \frac{2\,M}{2\,EI} \cdot \frac{dM}{dP} \cdot ds$$

$$= \frac{M}{EI} \cdot \frac{dM}{dP} \cdot ds$$

Now $\frac{dW}{dP}$ = element of deflection in direction of W caused by movement of element ds

$\frac{dM}{dP}$ – bending moment caused by unit load

$= m$

∴ Total deflection in given direction

$$= \int \frac{M\,m}{E\,I}\,d\,s. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

In the case in which the unit force is horizontal we have $m = 1 \times y$, Fig. 54.

$$\therefore \text{Horizontal deflection} = c\,c' = \int_A^C \frac{M\,y}{E\,I}\,d\,s \quad \ldots\ldots\ldots \quad (3a).$$

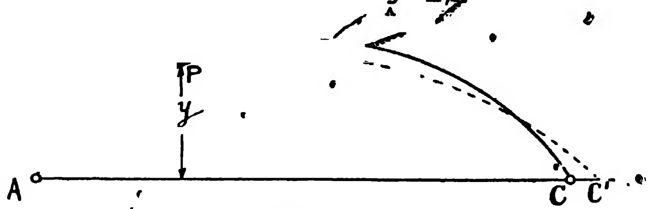


Fig. 54.

Similarly, for vertical deflections, if x is the horizontal distance from the point, where the unit vertical load is applied, to the point at which the bending moment is being considered, we shall get vertical deflection

$$\int \frac{M\,x\,d\,s}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3b).$$

Since in the two-pinned arch the horizontal deflection of the end c must be zero, *i.e.* the length of the span must not change,

we get
$$\int_A^C \frac{M\,y\,d\,s}{E\,I} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Now for an arch $M = M_1 - H\,.\,y$ .... (5)

where M_1 is the bending moment for the given load on a freely-supported beam of span equal to that of the arch, and H is the horizontal thrust required.

$$\therefore \text{ We have } \int^C \frac{(M_1 - H\,y)\,y\,d\,s}{E\,I} = 0$$

$$H \int^C \frac{y^2\,d\,s}{E\,I} = \int^C \frac{M_1\,y\,d\,s}{E\,I}$$

$$\therefore\ H = \frac{\int_A^C \frac{M_1 y\, ds}{EI}}{\int_A^C \frac{y^2\, ds}{EI}} \quad \dots\dots\dots\dots\dots\dots (6)$$

If, as is usual, E is constant throughout the arch, this becomes

$$H = \frac{\int_A^C \frac{M_1 y\, ds}{I}}{\int_A^C \frac{y^2\, ds}{I}} \quad \dots\dots\dots\dots\dots\dots (7)$$

When the form of the arch is such that these integrations cannot be performed easily, we take finite short equal lengths δs for ds, and if I is constant an equation becomes

$$H = \frac{\sum_A^C M_1 y}{\sum_A^C y^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots (8)$$

and since $M = H_0 z$ where H_0 is any trial value of the thrust, and z is the ordinate of the resulting line of pressure, we get the same result as equation (1), viz.,

$$H = \frac{H_0 \Sigma y z}{\Sigma y^2}$$

Reaction Locus.—When dealing with isolated loads—and any load system can always be considered as divided up into isolated loads—the simplest procedure for obtaining the reactions is by means of the 'Reaction Locus.' In practice, the load is usually distributed to the arch by a number of vertical columns, so that even with a uniformly distributed load on the span, the load on the arch itself is concentrated at a number of points.

The 'Reaction Locus' is a line which gives the point of intersection of the two reactions for any position of an isolated load.

If, for instance, in Fig. 55, K M J is the reaction locus, then, if an isolated P be placed at the point F, and the vertical through F intersects the reaction locus at M, the lines M A, M C obtained

by joining the point M to each of the hinges give the directions of the resultant reactions R_A and R_c at the hinges. This gives us the line of pressure A M C at once, and the bending moment at any point is equal to the product of the horizontal thrust H and the vertical distance between the line of pressure and the centre line of the arch at the point. If the reaction locus is known, and h_F is its height at the point F, then the value of the horizontal thrust H is readily calculated as follows:—

The Δ M A N must represent to some scale the Δ of forces at the point A.

$$\therefore \frac{V_A}{H} = \frac{MN}{NA} = \frac{h_F}{aL}$$

$$\therefore H = \frac{V_A aL}{h_F} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

and $V_A = P(1 - a)$.

$$\therefore H = \frac{Pa(1-a)L}{h_F} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

At the point F the bending moment will be equal to H $\times$ M F.

$$i.e. \text{ B.M. at F} = H(h_F - y_F)$$

$$= Pa(1-a)L\left\{1 - \frac{y_F}{h_F}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(11)$$

Parabolic Arch Rib.—In the case of a parabolic arch, the equation to the reaction locus can be readily calculated on certain assumptions.

The most important assumption is that the section of the rib is slightly greater at the springings than at the crown.

Referring back to equation (7) we see that the general equation, if the ordinate of the B.M. diagram is z (corresponding to H = 1 in equation (1)), is

$$H = \frac{\int_A^C \frac{y z \, ds}{I}}{\int_A^C \frac{y^2 \, ds}{I}}$$

If now the moment of inertia I of the arch rib is assumed to vary as the secant of the angle of inclination θ at any point, $\frac{ds}{I} = \frac{dx}{I_0}$ where I_0 is the moment of inertia at the crown, because

$$dx = ds \cos\theta \text{ and } I = I_0 \sec\theta$$

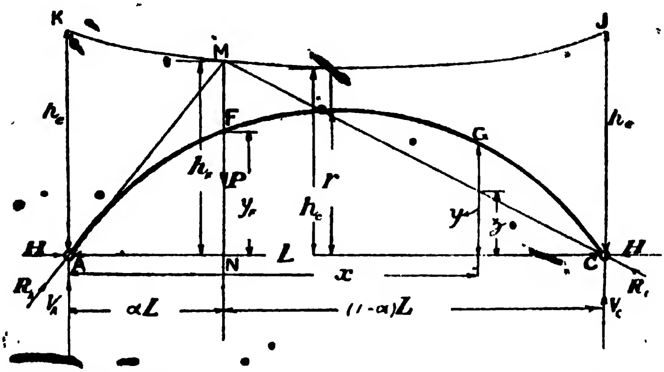


Fig. 55.—Reaction Locus for two-pinned Parabolic Arch.

Our equation then becomes

$$H = \frac{\int_A^C yz\,dx}{\int_A^C y^2\,dx} \quad \text{.........................(12)}$$

It can then be shown that

$$h_r = \frac{1{\cdot}6\,r}{1 + a - a^2} \quad \text{.............................(13)}$$

$$H = \frac{5}{8}\,\frac{P L a(1 - a)(1 + a - a^2)}{r} \quad \text{......(14)}$$

This gives the following values of h_r and H in terms of r and $\frac{PL}{r}$ respectively.

a	h_r (Multiply by r)	H (Multiply by $\frac{PL}{r}$)	
0	1·60	0	1·0
·1	1·47	·061	·9
·2	1·38	·116	·8
·3	1·32	·159	·7
·4	1·29	·186	·6
·5	1·28	·195	·5
	h_r	H	a

From these values the reaction locus K M J can be drawn, and by resolving the reactions for any series of loads into horizontal and vertical components and adding, we can get the total horizontal aud vertical reactions, and thus will be able to draw the line of pressure, or calculate the bending moment at any point.

PROOF OF FORMULÆ.—Although the integrations required to prove equations (13) and (14) are what are regarded as simple ones, we believe that many engineers find some difficulty in inserting the steps which are usually omitted, and so we give them at full length.

The general equation to a parabola is

$$y = a + bx + cx^2 \dots\dots\dots\dots\dots\dots (15)$$

In our case, Fig. 55, at A, where $x = 0, y = 0 \therefore a = 0.$ Again, at the centre where

$$x = \frac{L}{2}, y = r$$

$$\therefore r = \frac{bL}{2} + \frac{cL^2}{4} \dots\dots\dots\dots\dots\dots (16)$$

Finally, at the end C, where $x = L, y = 0$

$$\therefore 0 = bL + cL^2, \textit{i.e. } b = -cL \text{ and } a = 0$$

$\therefore$ in (15)

$$r = -\frac{cL^2}{2} + \frac{cL^2}{4} = -\frac{cL^2}{4}$$

$$\left.\begin{aligned} \therefore c &= -\frac{4r}{L^2} \\ b &= \frac{4r}{L} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Hence our parabola is given by

$$y = \frac{4r}{L^2}(Lx - x^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

The 'arch-square sum' is given by

$$AS = \int y^2\,dx = \frac{16r^2}{L^4}\int_0^L (Lx - x^2)^2\,dx \quad \ldots\ldots\ldots \quad (19)$$

Now
$$\int_0^L (Lx - x^2)^2\,dx = \int_0^L x^2(L - x)^2\,dx$$

$$= \int_0^L x^2(L^2 - 2Lx + x^2)\,dx$$

$$= \left[\frac{L^2x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^L$$

$$= \left[\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right] - 0$$

$$= \frac{L^5}{30}\Big[10 - 15 + 6\Big] = \frac{L^5}{30}$$

$$AS = \frac{8r^2L}{15} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

The 'load-arch sum' is given by

$$LA = \int_0^L yz\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

The value of z changes abruptly at M.
From A to N it is given by

$$z = V_A\,x = P(1 - a)\,x$$

From N to C it is given by

$$z = V_C(L - x) = Pa(L - x)$$

Our integration therefore has to be performed in two steps and equation (21) becomes

$$L\,A = \int_0^L \frac{4\,r\,x}{L^2}(L - x)\,P\,(1 - a)\,x\,dx$$
$$+ \int_{aL}^{L} \frac{4\,r\,x}{L^2}(L - x)\,P\,a\,(L - x)\,dx \;\ldots\ldots\; (22)$$

$$= \frac{4\,P\,r}{L^2}\left\{\int_0^L (1 - a)\,x^2\,(L - x)\,dx + \int_{aL}^{L} a\,x\,(L - x)^2\,dx\right\}$$

The first integral comes equal to

$$(1 - a)\left[\frac{x^3\,L}{3} - \frac{x^4}{4}\right]_0^{aL} = (1 - a)\left[\left(\frac{a^3\,L^4}{3} - \frac{a^4\,L^4}{4}\right) - 0\right]$$

$$= (1 - a)\,a\,L^4\left[\frac{a^2}{3} - \frac{a^3}{4}\right] \;\ldots\ldots\ldots\ldots\ldots\ldots\; (23)$$

The second integral comes equal to

$$a\int_{aL}^{L} (L^2\,x - 2\,L\,x^2 + x^3)\,dx$$

$$= a\left[\frac{L^2\,x^2}{2} - \frac{2\,L\,x^3}{3} + \frac{x^4}{4}\right]_{aL}^{L}$$

$$= a\left[\left\{\frac{L^4}{2} - \frac{2\,L^4}{3} + \frac{L^4}{4}\right\} - \left\{\frac{a^2\,L^4}{2} - \frac{2\,a^3\,L^4}{3} + \frac{a^4\,L^4}{4}\right\}\right]$$

$$= a\,L^4\left[\left\{\frac{6 - 8 + 3}{12}\right\} - a^2\left\{\frac{1}{2} - \frac{2\,a}{3} + \frac{a^2}{4}\right\}\right]$$

$$= \frac{a\,L^4}{12}\left[1 - 6\,a^2 + 8\,a^3 - 3\,a^4\right]$$

This factorises to

$$\frac{a\,L^4\,(1 - a)}{12}\left\{1 + a - 5\,a^2 + 3\,a^3\right\} \;\ldots\ldots\ldots\; (24)$$

Adding (23) and (24) we get

$$\text{L A} = \frac{(1 - a)\,a\,\text{L}^4}{12}\left[4a^2 - 3a^3 + 1 + a - 5a^2 + 3a^3\right] \times \frac{4\,\text{P}r}{\text{L}^2} \quad \ldots\ldots\ldots (25)$$

$$\frac{(1 - a)\,a\,\text{L}^2}{3}\left[1 + a - a^2\right]\text{P}r$$

$$\therefore\ \text{H} = \frac{\text{L A}}{\text{A S}} = \frac{\text{P}r\,a\,(1 - a)\,\text{L}^2\left[1 + a - a^2\right]}{3 \times \frac{8r^2\text{L}}{15}}$$

$$= \frac{5\,\text{P L}\,a}{8r}(1 + a - a^2)(1 - a) \quad \ldots\ldots\ldots\ldots (26)$$

From equation (10):

$$h_{\text{F}} = \frac{\text{P}\,a\,(1 - a)\,\text{L}}{\text{H}}$$

$$= \frac{\text{P}\,a\,(1 - a)\,\text{L}}{\frac{5\,\text{P L}\,a\,(1 + a - a^2)(1 - a)}{8r}}$$

$$= \frac{8r}{5(1 + a - a^2)}$$

$$= \frac{1{\cdot}6\,r}{(1 + a - a^2)}$$

On comparing this with equation (13) we see that this is the result required.

Uniform Load over Whole Span.—As an interesting application of the above formulæ we will assume that a load of intensity p covers the span.

Consider a short length of the span, we may consider the load P acting at its centre as $p\,.\,d\,(a\,\text{L}) = p\,\text{L}\,d\,a$.

Then due to this load we have a thrust

$$= d\,\text{H} = \frac{5\,p\,\text{L}}{8r}\,a\,(1 - a)(1 + a - a^2)\,d\,a$$

$$\therefore\ \text{Total thrust} = \text{H} = \frac{5\,p\,\text{L}^2}{8r}\int_0^1 a\,(1 - a)(1 + a - a^2)\,d\,a$$

$$\frac{5\,p\,L^2}{8\,r}\int^{1}(a - 2\,a^3 + a^4)\,d\,a$$

$$= \frac{5\,p\,L^2}{8\,r}\left[\frac{a^2}{2} - \frac{2\,a^4}{4} + \frac{a^5}{5}\right]_0^1$$

$$= \frac{5\,p\,L^2}{8\,r}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{5} - 0\right]$$

$$= \frac{p\,L^2}{8\,r}$$

This is the well-known result, because with a parabolic arch carrying a load uniformly distributed along the horizontal, the line of pressure coincides with the centre line of the arch and the above result follows at once.

Uniform Load over Half Span.—If a uniform load covers half the span from one springing to the crown, a similar treatment gives

$$H = \frac{5\,p\,L^2}{8\,r}\int_0^{\frac{1}{2}}(a - 2\,a^3 + a^4)\,d\,a$$

$$= \frac{5\,p\,L^2}{8\,r}\left[\frac{a^2}{2} - \frac{2\,a^4}{4} - \frac{a^5}{5}\right]_0^{\frac{1}{2}}$$

$$= \frac{5\,p\,L^2}{8\,r}\left[\frac{1}{8} - \frac{1}{32} + \frac{1}{160}\right]$$

$$= \frac{p\,L^2}{16\,r}$$

This again is the well-known result obtained from the previous one from considerations of symmetry.

Temperature Thrust for Parabolic Arch.—If β is coefficient of linear expansion for the arch and t is the rise in temperature, the span, if free to expand, will become of length $L\,(1 + \beta\,t)$; $L\,\beta\,t$ is therefore the horizontal deflection in resisting which the temperature thrust H_T is induced.

Now from equation (3 a)

$$L\,\beta\,t = \int^{C}\frac{M\,y\,d\,s}{E\,I}$$

And since H_T is the only force acting in this case, $M = H_T y$

$$\therefore L \beta t = \int_A^C \frac{H_T y^2 \, ds}{E I} \quad \text{.......................(23)}$$

In the case of the parabolic arch rib with the previous assumption as to the variation of I, we get

$$L \beta t = \frac{H_T}{E I_0} \int y^2 \, dx$$

$$= \frac{H_T}{E I_0} \cdot \frac{8 r^2 L}{15}$$

$$\therefore \quad H_T = \frac{15 E I_0 \beta t}{8 r^2} \quad \text{..........................(24)}$$

For mild steel we may take

$$\beta = 6{\cdot}7 \times 10^{-6} \text{ per } {}^\circ\text{F}$$

For concrete

$$\beta = 6{\cdot}0 \times 10^{-6} \text{ per } {}^\circ\text{F}.$$

Taking for steel $E = 30 \times 10^6$ lbs. per sq. in.

$$H_T \text{ for steel} = \frac{375 \, I_0 t}{r^2} \quad \text{...........................(25)}$$

In this formula both I_0 and r must be taken in inches, and H_T will be in lbs.

NUMERICAL EXAMPLE.—*Take the case of a two-hinged parabolic arch of* 120 *ft. span, and rise* 20 *ft. with a live load of* 20,000 *lbs. per panel, and a dead load of* 10,000 *lbs. per panel, the load being distributed at* 10 *points, as shown* (*Fig.* 56).

We will first tabulate values of H at each point per unit load ; by symmetry we need only consider one-half of the span.

We then get :—

Point.	y (ft.)	h (ft.)	H (for unit load).
1	7·2	29·4	·368
	12·8	27·6	·696
3	16·8	26·4	·953
4	19·2	25·8	1·116
5	20	25·6	1·172

Now take the live load on the left-hand half of the span, and the

dead load over the whole span, and find the thrust for the load at each point. We then get :—

Point.	Load (ten thousands of pounds).	H (in ten thousands of pounds).
1	3	3 × ·368 = 1·104
2	3	3 × ·696 = 2·088
3	3	3 × ·953 = 2·859
4	3	3 × 1·116 = 3·348
5	2	2 × 1·172 = 2·344
6	1	= 1·116
7	1	= ·953
8	1	= ·696
9	1	= ·368
	Total ...	= 14·878

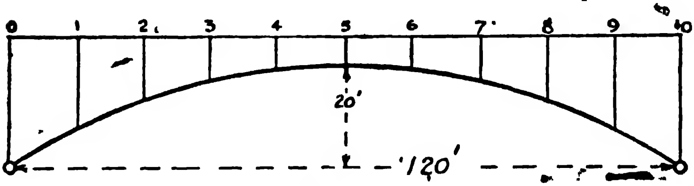


Fig. 56.

Now calculate the B.M. at ¼ and ¾ spans.

The vertical reaction at the left-hand end—

$$= \frac{10 \times 10{,}000}{} + \frac{5 \times 20{,}000 \times 3}{} - 15{,}000 = 110{,}000 \text{ lbs.}$$

∴ B.M. due to load at ¼ span

$= 110{,}000 \times 30 - 30{,}000\,(18 + 6) = 2{,}580{,}000$ ft. lbs.

At ¾ span working from other end where reaction $= 70{,}000$.

∴ B.M. due to load

$= 70{,}000 \times 30 - 10{,}000\,(18 + 6) = 1{,}860{,}000$ ft. lbs.

At ¼ and ¾ span $y = 15$ ft.

∴ B.M. due to thrust $= 148{,}780 \times 15 = 2{,}231{,}700$ ft. lbs.

Positive B.M. $= 2{,}580{,}000 - 2{,}231{,}700 = 348{,}300$ ft. lbs.

Negative B.M. $= 2{,}231{,}700 - 1{,}860{,}000 = 371{,}700$ ft. lbs.

***Formulæ for Circular Arch Rib.**—We will next consider the case of a circular arch rib A G C, Fig. 57, of constant cross-section and radius R.

Let θ be the half-angle subtended by the arch rib at the centre, and let a load P be taken at a point F subtending an angle ϕ at the centre line; in order to simplify the integrations to calculate H we will assume an equal load P to be placed symmetrically on the other side; the value of H for one load will then be one-half that for the two loads.

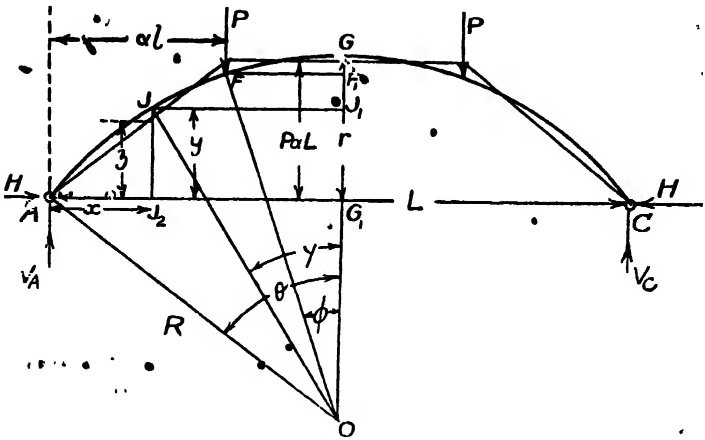


Fig. 57.—Circular Arch Rib.

Then working in polar co-ordinates, for the point G

$$x = AG_1 - JJ_1$$
$$= R\sin\theta - R\sin\gamma$$
$$= R(\sin\theta - \sin\gamma) \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$y = OJ_1 - OG_1$$
$$= R\cos\gamma - R\cos\theta$$
$$= R(\cos\gamma - \cos\theta) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$V_A = V_c = P$

$\therefore$ For $\gamma < \phi$, z = 'free' B.M. at J = $P\,a\,L$, because the B.M. is constant between P P.

and $a\,L = R(\sin\theta - \sin\phi) \quad \ldots\ldots\ldots\ldots\ldots (3)$

i.e., $a\,L$ = value of x above for $\gamma = \phi$

i.e., $z = P\,R(\sin\theta - \sin\phi) \quad \ldots\ldots\ldots\ldots\ldots (4)$

For $\gamma > \phi$, $z = P\,.\,x$

$P\,R(\sin\theta - \sin\gamma) \quad (5)$

Also $d s = R\, d \gamma$

We will in each case work the integral from the upper limit :

$$\therefore \int^{C} y^2\, d s = 2 \int_{0}^{\theta} R^2 (\cos \gamma - \cos \theta)^2 R\, d \gamma$$

$$= 2 R^3 \int_{0}^{\theta} (\cos^2 \gamma - 2 \cos \gamma \cos \theta + \cos^2 \theta)\, d \gamma$$

$$= 2 R^3 \left(\int \cos^2 \gamma\, d \gamma - 2 \cos \theta \int \cos \gamma\, d \gamma + \cos^2 \theta \int d \gamma \right)$$

$$= 2 R^3 \left(\cos^2 \theta \,.\, \theta - 2 \cos \theta \sin \theta + \int_{0}^{\theta} \frac{1 + \cos^2 \gamma}{2}\, d \gamma \right)$$

$$= 2 R^3 \left(\theta \cos^2 \theta - 2 \cos \theta \sin \theta + \frac{\theta}{2} + \frac{\sin 2 \theta}{4} \right)$$

$$= \frac{R^3}{2} \left(4 \theta \cos^2 \theta - 4 \sin 2 \theta + 2 \theta + \sin 2 \theta \right)$$

$$= \frac{R^3}{2} \left(4 \theta \cos^2 \theta + 2 \theta - 3 \sin 2 \theta \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\int_{A}^{C} y\, z\, d s = 2 \left\{ \int_{0}^{\phi} R^3 P (\sin \theta - \sin \phi) (\cos \gamma - \cos \theta)\, d \gamma \right.$$

$$\left. + \int_{\phi}^{\theta} R^3 P (\sin \theta - \sin \gamma) (\cos \gamma - \cos \theta)\, d \gamma \right\} \quad (7)$$

The first integral is equal to

$$R^3 P (\sin \theta - \sin \phi) \int_{0}^{\phi} (\cos \gamma - \cos \theta)\, d \gamma$$

$$= R^3 P (\sin \theta - \sin \phi) \Big[\sin \gamma - \gamma \cos \theta \Big]_{0}^{\phi}$$

$$= R^3 P \left(\sin \theta - \sin \phi \right) \left(\sin \phi - \phi \cos \theta \right)$$

$$= R^3 P \left(\sin \theta \sin \phi - \sin^2 \phi - \frac{\phi \sin 2 \theta}{2} + \phi \sin \phi \cos \theta \right) \quad (8)$$

The second integral is equal to

$$R^3 P \int_{\phi}^{\theta} \{\sin\theta(\cos\gamma - \cos\theta) - \sin\gamma\cos\gamma + \cos\theta\sin\gamma\}\, d\gamma$$

$$= R^3 P \left[\sin\theta \int_{\phi}^{\theta} (\cos\gamma - \cos\theta)\, d\gamma - \int_{\phi}^{\theta} \frac{\sin 2\gamma}{2}\, d\gamma + \cos\theta \int_{\phi}^{\theta} \sin\gamma\, d\gamma \right]$$

$$= R^3 P \left[\sin\theta(\sin\gamma - \gamma\cos\theta) + \frac{\cos 2\gamma}{4} - \cos\theta\cos\gamma \right]_{\phi}^{\theta}$$

$$= R^3 P \left[\sin^2\theta - \frac{\theta\sin 2\theta}{2} + \frac{\cos 2\theta}{4} - \cos^2\theta - \sin\theta\sin\phi + \frac{\phi\sin 2\theta}{2} - \frac{\cos 2\phi}{4} + \cos\theta\cos\phi \right] \quad \text{.........(9)}$$

Adding (8) and (9) we get

$$\int_{A}^{C} y z\, ds = 2 R^3 P \left[\sin^2\theta - \sin^2\phi - \frac{\theta\sin 2\theta}{2} + \frac{\cos 2\theta}{4} - \cos^2\theta - \frac{\cos 2\phi}{4} + \cos\theta\cos\phi + \phi\sin\phi\cos\theta \right]$$

$$= R^3 P \left[\sin^2\theta - \sin^2\phi - 2\cos\theta(\cos\theta - \cos\phi + \theta\sin\theta - \phi\sin\phi) \right] \quad \text{......(10)}$$

(To follow the last step we note that

$$\frac{\cos 2\theta}{4} - \frac{\cos 2\phi}{4} = \frac{\sin^2\phi}{2} - \frac{\sin^2\theta}{2} \Big)$$

$$\text{and } H = \frac{1}{2} \frac{\int_{A}^{C} y z\, ds}{\int_{A}^{C} y^2\, ds}$$

$$= \frac{P[\sin^2\theta - \sin^2\phi - 2\cos\theta(\cos\theta - \cos\phi + \theta\sin\theta - \phi\sin\phi)]}{(4\theta\cos^2\theta + 2\theta - 3\sin 2\theta)} \quad \text{......(11)}$$

The above is the thrust for a single isolated load, the $\frac{1}{2}$ being introduced as above explained.

Special Case of Semicircular Arch.—In the extreme case of a semicircular arch, for which $\theta = 90^\circ$ or $\frac{\pi}{2}$, equation (11) gives

$$H = \frac{P\cos^2\phi}{\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

For load P at the crown, where $\phi = 0$, this gives

$$H = \frac{P}{\pi} = \cdot 318 \text{ P nearly}$$

From the table on page 126 for a parabolic arch, we get for the crown $H = \frac{\cdot 195 \text{ P L}}{r}$, and for $r = \frac{L}{2}$, which will give the same height of crown as the semicircular arch, $H = \cdot 390$ P.

Special Case of 90° Circular Arch.—If the angle subtended at the centre of a circular arch is 90°, $R = \frac{L}{\sqrt{2}}$ and the rise r will be equal to $\frac{L}{\sqrt{2}} - \frac{L}{2} = \cdot 207$ L.

This is nearly one-fifth of the span, and is a rise fairly common in practice.

In this case $\theta = 45^\circ = \frac{\pi}{4}$.

$$\therefore \text{ The 'arch-square sum'} = \frac{R^3}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} - 3\right)$$

$$= \frac{L^3}{4\sqrt{2}} \times \cdot 1416$$

$$= \cdot 02503 \text{ L}^3$$

$$\text{or} \quad = \cdot 0708 \text{ R}^3$$

We will tabulate as before values of H for 8 equal divisions of the arch, *i.e.*, 4 values of ϕ.

Angle ϕ.	Value of H in terms of P.	Angle ϕ.	Value of H in terms of P.
0°	·909	$33\frac{3}{4}$°	·315
$11\frac{1}{4}$°	·835	45°	0·000
$22\frac{1}{2}$°	·607		

These values are plotted against divisions of the arch in Fig. 58.

Special Case of 60° Circular Arch.—If the angle subtended at the centre of a circular arch is 60°, the rise will be equal to $L\left(1 - \frac{\sqrt{3}}{2}\right) = \cdot134\ L$. This is between $\frac{1}{7}$ and $\frac{1}{8}$ of the span, and would give what is known as a flat arch.

In this case $\theta = 30° = \frac{\pi}{6}$ and $R = L$.

∴ The 'arch-square sum' = (from equation (6))

$$\frac{R^3}{2}\left\{\frac{4\pi}{6}\ \frac{3}{4} + 2\pi\ \ \frac{3\sqrt{3}}{2}\right\}$$

$$= \cdot00996\ R^3 \quad \ldots\ldots\ldots\ldots\ldots (7)$$

$$= \cdot00996\ L^3 \quad \ldots\ldots\ldots\ldots\ldots (7a)$$

The values of H for angles ϕ equal to 0, 15°, 30°, and 45° can then be calculated by equation (11) and tabulated as follows, but as they are difference formulæ some care has to be taken to get accuracy.

Angle ϕ.	Value of H in terms of P.	Angle ϕ.	Value of H in terms of P.
0°	1·44	$22\frac{1}{2}°$	·53
$7\frac{1}{2}°$	1·31	30°	0·00
15°	1·00		

These values are plotted in Fig. 58 upon the same base as for the 60° arch.

Temperature Thrust for Circular Arch.—By equation (23), p. 131—

$$L\beta t = \int^{c} \frac{H_T y^2\, ds}{EI}$$

$$H_T = \frac{EIL\beta t}{\text{arch-square sum}}$$

For the 60° arch—

$$H_T = \frac{E\,I\,L\,\beta\,t}{\cdot 00996\,L^3}$$

∴ for steel, taking $E = 30 \times 10^6$ and $\beta = 6{\cdot}7 \times 10^{-6}$

$$H_T = \frac{20{,}200\,I\,t}{L^2}$$

For concrete, taking $E = 2 \times 10^6 \times \beta = 6{\cdot}0$

$$H_T = \frac{1200\,I\,t}{L^2}$$

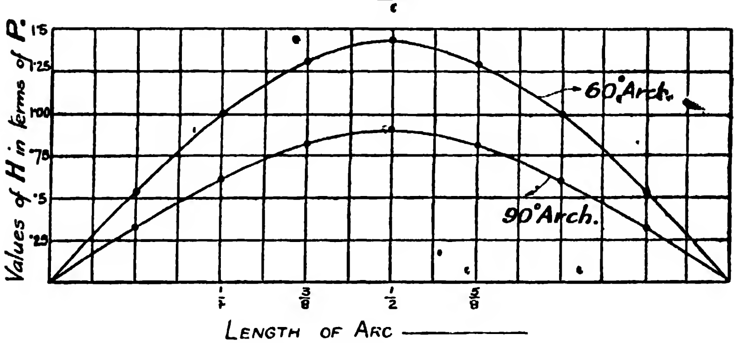


Fig. 58.—Thrusts for Circular Arch Ribs.

For the 90° arch—

$$H_T = \frac{E\,I\,\beta\,t}{\cdot 02503\,L^2}$$

$$= \frac{8040\,I\,t}{L^2} \text{ for steel}$$

$$= \frac{480\,I\,t}{L^2} \text{ for concrete}$$

Reaction Loci for Circular two-pinned Arch Ribs.—We will now find the reaction loci for the circular two-pinned arches that we have considered.

Semicircular Arch.—Let a load P, Fig. 59, be applied at a point F on a semi-circular arch; then

$$\frac{MN}{AN} = \frac{V_A}{H} \qquad \text{.........................}(13)$$

Now $V_A = P \times \frac{NC}{AC} = \frac{P \times R(1 + \sin\phi)}{2R}$

And $H =$ (from equation (11), p. 135, putting $\theta = \frac{\pi}{2}$)

$$\frac{P\{1 - \sin^2\phi\}}{\pi}$$

Also $\quad AN = R(1 - \sin\phi)$

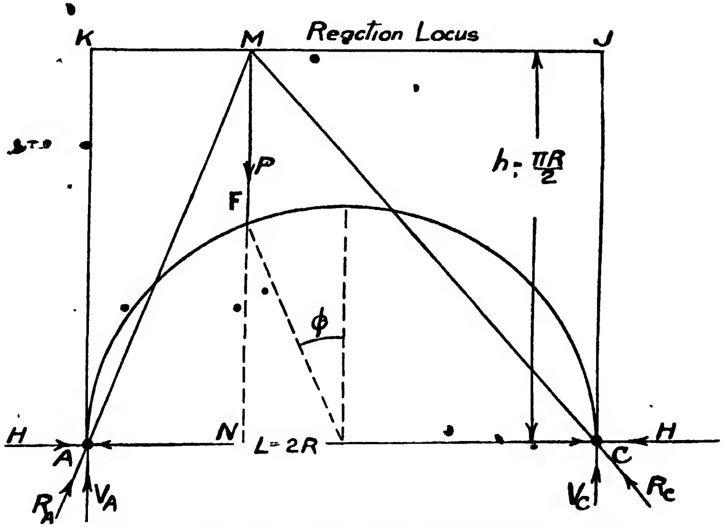


Fig. 59.—Reaction Locus for Semicircular Arch.

∴ From (13) MN

$$= \frac{R(1 - \sin\phi) \times PR(1 + \sin\phi)}{2R} \div \frac{P\{1 - \sin^2\phi\}}{\pi}$$

$$= h = \frac{\pi R}{2} \quad \text{.................................} \quad (14)$$

Therefore the reaction locus in this case is a straight line KMJ at height $\frac{\pi R}{2}$ above AC.

90° *Circular Arch.*—In this case, Fig. 60,

$$L = R\sqrt{2}$$

$$AN = AQ - NQ = AQ - FS$$

$$= \frac{L}{2} - R \sin \phi$$

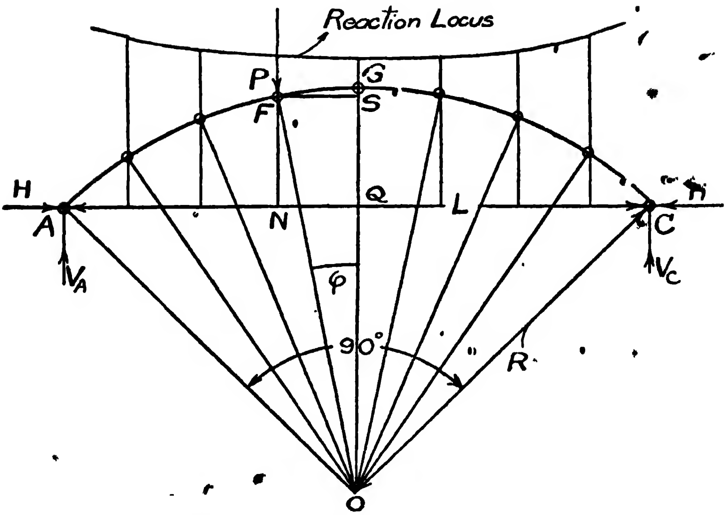


Fig. 60.—*Reaction Locus for* 90° *Arch.*

And $$V. = \frac{P \times NC}{AC} = \frac{P\left(\frac{L}{2} + R \sin \phi\right)}{L}$$

$$\therefore h \quad \frac{V_A \cdot AN}{H} = \frac{P\left(\frac{L}{2} + R \sin \phi\right)\left(\frac{L}{2} - R \sin \phi\right)}{LH}$$

$$= \frac{P\left(\frac{L^2}{4} - R^2 \sin^2 \phi\right)}{LH} = \frac{PL^2}{2LH}\left(\frac{1}{2} - \frac{R^2}{2L^2} \sin^2 \phi\right)$$

$$= \frac{PL}{2H}\left(\frac{1}{2} - \sin^2 \phi\right)$$

Taking the values tabulated on p. 136, we get the following results.

Angle ϕ	h	Angle ϕ	h
0°	·276 L	22½°	·291 L
11¼°	·277 L	33¾°	·303 L

This gives the reaction locus shown in Fig. 60.

60° *Circular Arch.*—In this case, Fig. 61, as in the previous case we shall have

$$h = \frac{P\left(\frac{L^2}{4} - R^2 \sin^2 \phi\right)}{L\,H},$$

in the case R = L.

$$\therefore\ h = \frac{P\,L\left(\frac{1}{4} - \sin^2 \phi\right)}{H}$$

Taking the values tabulated on p. 137 this gives—

Angle ϕ	h	Angle ϕ	h
0°	·174 L	15°	·183 L
7½°	·178 L	22½°	·188 L

This gives the reaction locus shown in Fig. 61.

Procedure for Design.—In designing a two-pinned arch rib to carry a given load we do not know the sections accurately to start with, and so cannot make an accurate computation of the stresses; in this case a good method of procedure is as follows.

First assume that the arch is parabolic and that the section varies in the manner set out on p. 124. With this assumption the maximum thrusts (including temperature thrust) and bending moments, and thus the stresses at the various points can be

found; the necessary section at various points can then be designed.

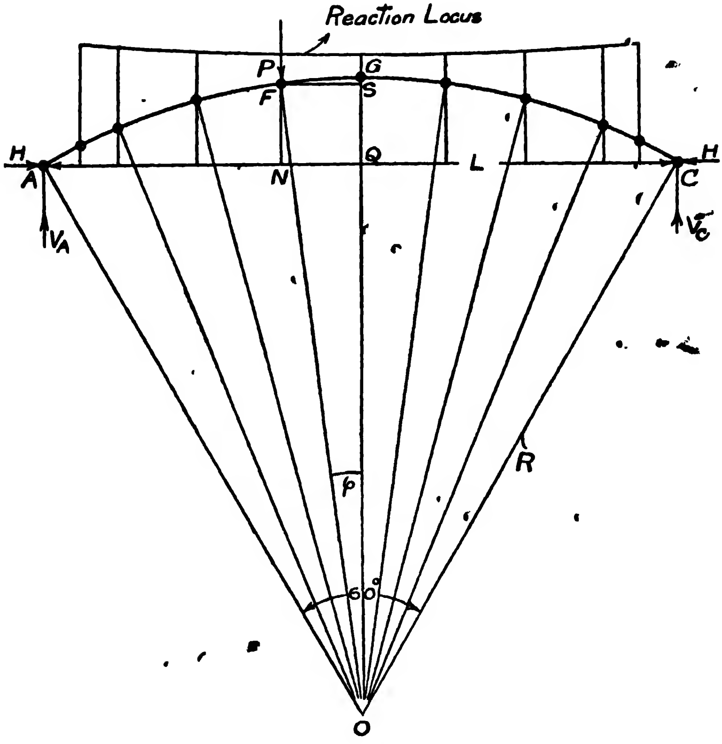


Fig. 61.—*Reaction Locus for* 60° *Arch.*

An accurate determination of the thrusts can then be found for the section so designed by means of the formulæ

$$H = \frac{\Sigma \frac{M_1 y}{I}}{\Sigma \frac{y^2}{I}}$$

$$H_T = \frac{E\, I\, n\, \beta\, t}{\Sigma \frac{y^2}{I}}$$

where n is the number of elements of the arch that are considered.

In this way corrected stresses are calculated, and, if necessary, modifications in the section can be made.

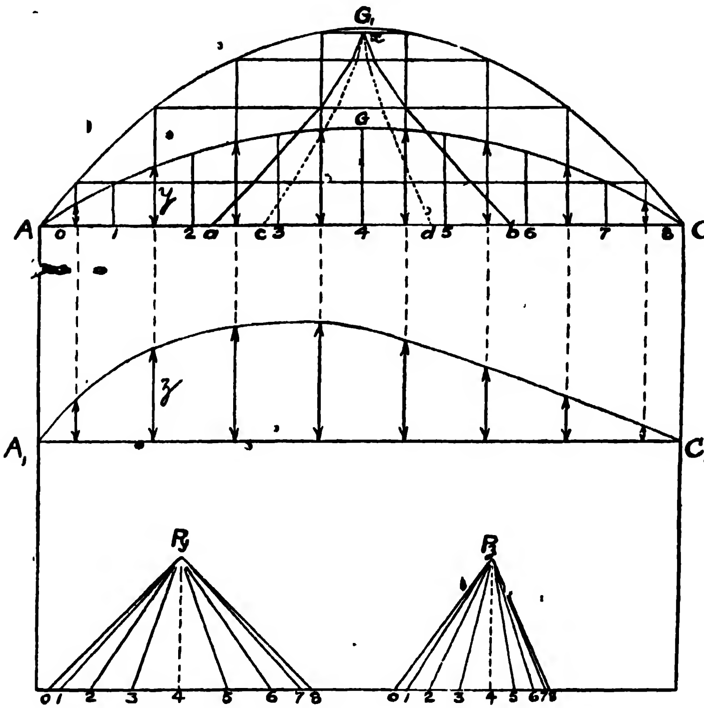


Fig. 62.—Graphical Construction for Arch Summations.

Graphical Constructions for Arch-square and Load-arch Sums.—In cases where we cannot obtain sufficient accuracy by working by the formulæ for parabolic and circular arch ribs, the following graphical constructions may be employed. Draw the arch centre-line A G C, Fig. 62, and divide *it* (not the span) into a convenient number of equal parts; to prevent the figure from becoming too crowded we have shown only eight

parts, but more than this should be taken to ensure accuracy. Ordinates are then drawn through the mid-points of these elements, and, in order to facilitate the construction, these mid-ordinates are drawn to an enlarged vertical scale (double scale in the figure). Horizontal lines are drawn through the mid-points on the enlarged scale and are treated as force lines.

The B.M. diagram $A_1 C_1$ for the given loading is then drawn, and the corresponding mid-ordinates are marked and represent the z's.

Now take horizontal base-lines, and to a convenient scale of reduction set out the ordinates y, z respectively at points 0, 1, 2, &c.

Above the points 4 choose poles Py, Pz, and draw vector figures at the same polar distance p, and, starting at the point x, draw link-polygons for the y's and z's, the intercepts upon the base A C being a, b and c, d respectively. Then if H_0 is the trial thrust corresponding to which the B.M. diagram is drawn, the thrust H that we require to find will be given by:

$$H = H_0 \times \frac{cd}{ab}$$

Proof. $\Sigma y^2 = \Sigma (y \times y) =$ sum of moments about the base line A C of a number of horizontal forces equal to y acting at the top of the mid-ordinates. According to the property of link polygons, the moment of a given force system about any point is equal to the polar distance multiplied by the intercept between the first and last links upon a line drawn through the point parallel to the resultant force.* This intercept in this case is ab.

$$\therefore \Sigma y^2 = p \times ab$$

$\Sigma yz =$ Sum of moments about the base line A C of a number of horizontal forces equal to z acting at the top of the mid-ordinates, and so

$$\Sigma yz = p \times cd$$

$$\therefore H = \frac{H_0 \Sigma yz}{\Sigma y^2} = \frac{H_0 . cd}{ab}$$

The construction so far is for the case where the cross-section

* For proof see A, pp. 53, 61.

of the arch is constant throughout; if this is not the case we set out $\frac{y}{I}$ and $\frac{z}{I}$ upon the lines Y Y and Z Z instead of y and z respectively and then proceed as before.

When the temperature thrust is required, we have

$$H_T = \frac{E\,I\,L\,\beta\,t}{\Sigma y^2.\,\delta s}$$

If the number of elements into which we have divided our arch is n, then $\frac{L}{\delta s} = n$. So that the above equation becomes

$$H_T = \frac{E\,I\,n\,\beta\,t}{p \times a\,b}$$

I., of course, values of $\frac{y}{I}$ were set out in obtaining $a\,b$, I disappears from the formulæ.

Two-pinned Arch Ribs with Tie-rods.—Sometime two-hinged arch ribs are provided with tie-rods between the springings, a familiar example of this occurring in the arched roof of St. Pancras Station, London, the tie-rods in this case passing below the platform level.

In this case, instead of equating to zero the total vertical movement of the support C, we must equate this movement to the extension in the bar which the force H will cause, because if the abutments are replaced by a tie-rod, the thrust upon the arch is accompanied by a tension in the tie-bar.

∴ From equation (3*a*), p. 122, we get

$$\int^{C} \frac{M\,y\,d\,s}{E\,I} = \frac{H\,L}{E\,A} \quad \dots\dots\dots\dots (1)$$

where L = the length of the tie-rod

A = its cross-sectional area.

Putting as before $M = M_1 - H\,y$, we get

$$\int^{C} \frac{M_1\,y\,d\,s}{E\,I} - H\int^{C} \frac{y^2\,d\,s}{E\,I} = \frac{H\,L}{E\,A} \quad \dots\dots\dots\dots (2)$$

$$\therefore\ H = \frac{\int_A^C \frac{M_1 y\, d s}{E I}}{\int_A^C \frac{y^2 d s}{E I} + \frac{L}{E A}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

If E and I are constant throughout the span, this gives

$$H = \frac{\int_A^C M_1 y\, d s}{\int_A^C y^2 d s + \frac{L I}{A}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

If instead of integrating we take finite increments δs, we get, putting $M_1 = H_0 z$

$$H = \frac{H_0 \Sigma y z}{\Sigma y^2 + \frac{L I}{A \delta s}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Since $\frac{L I}{A \delta s}$ will always have a positive value, it is clear that H will be less in the present case than that in which the supports are unyielding; this is, of course, what one would expect, because the present case is an intermediate condition between the bent beam and the rigid arch.

CHAPTER VIII.

RIGID OR ELASTIC ARCHES (continued).

TWO-PINNED FRAMED ARCHES.

To determine the horizontal thrust in a two-pinned framed arch, such as a spandril arch, we first assume one of the ends, say C (Fig. 63), to be movable horizontally. If we apply a unit horizontal load at the point C, it follows from the treatment in

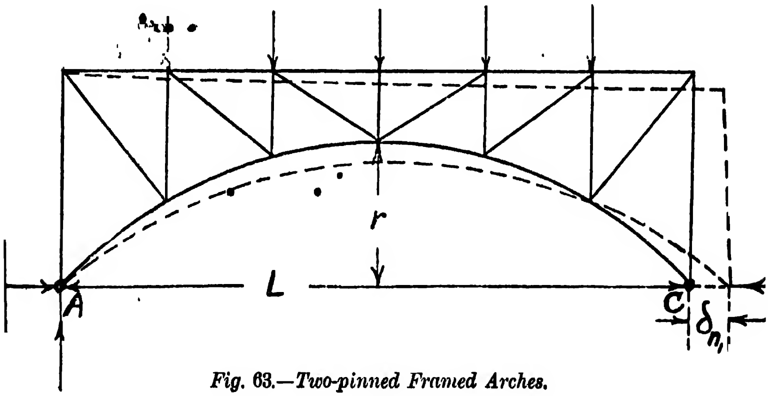


Fig. 63.—Two-pinned Framed Arches.

Chapter V. that the outward horizontal movement at C will be equal to

$$\delta_{n_1} = \Sigma \frac{F\,U\,L}{E\,A} \qquad \text{.........................(1)}$$

Where F = load or stress in any member due to given loading.
U = load or stress in that member due to unit horizontal load at C.
L = length of that member.
A = cross-sectional area of that member.
E = Young's modulus.

Also due to a single thrust H at the point c, the inward horizontal movement of the point c will be equal to

$$\delta_{n_2} = \text{H}\ \Sigma \frac{\text{U}^2\,\text{L}}{\text{E A}} \quad \text{.......................(2)}$$

because in this case F = H U in each member.

If the length of the span is not changeable, we must have $\delta_{n_1} = \delta_n$

$$\therefore\ \text{H} = \frac{\Sigma \frac{\text{F U L}}{\text{E A}}}{\Sigma \frac{\text{U}^2\,\text{L}}{\text{E A}}} \quad \text{.......................(3)}$$

$$= \frac{\text{Horizontal deflection of c due to given loading}}{\text{Horizontal deflection of c due to unit horizontal load at c}}$$

If E is constant throughout the structure this gives

$$\text{H} = \frac{\Sigma \frac{\text{F U L}}{\text{A}}}{\Sigma \frac{\text{U}^2\,\text{L}}{\text{A}}} \quad \text{.......................(4)}$$

The similarity between this formula and that for the arch rib will be apparent. Similarly for temperature thrust we shall have

$$\text{H}_\text{T} = \frac{\text{L}\,\beta\,t}{\Sigma \frac{\text{U}^2\,\text{L}}{\text{E A}}} \quad \text{.......................(5)}$$

Once we have determined the horizontal thrusts in this manner, the stresses in the arch can be found by the ordinary reciprocal figure construction, or by the method of moments. The procedure in this case is, therefore, as follows :—

1. Determine the stresses in the various members as if no horizontal thrusts occur.
2. (*a*) Calculate the horizontal movements at c due to the given loading and to a unit horizontal load at c, and divide the first by the second to give the thrust H.

   or (*b*) Find by displacement diagrams the horizontal deflections at c due to the loading and to a unit load at c, and divide the first by the second to give the thrust H.

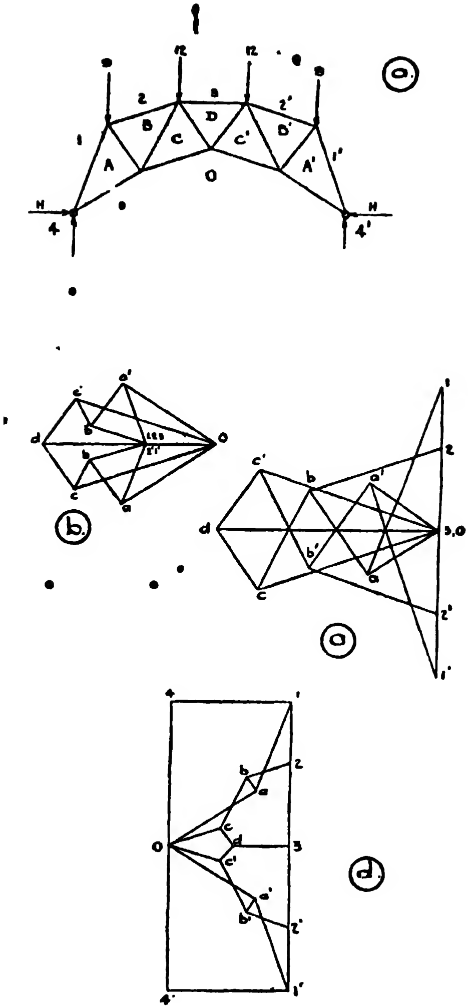


Fig. 63a.—Stress Diagrams for Arched Roof Truss.

3. Determine the stresses in the various members due to the load and the thrust H acting together.

Fig. 63*a* shows an example of this. (*a*) shows an arch roof truss hinged at the ends, the load being in tons.

The stress diagram (*c*) is first drawn for the vertical reactions only; this is equivalent to assuming one of the hinges replaced by a roller bearing. This diagram gives the value of F for each member.

Taking 10 tons as a unit horizontal load, the diagram (*b*) is next drawn, the load 01 being the only external load; in this diagram, of course, the points 1, 2, 3, 2′, 1 coincide. This diagram gives the value of U for each member. In these calculations the cross section of each of the members has to be assumed beforehand. The value of $\frac{\text{F U L}}{\text{A}}$ and $\frac{\text{U}^2\,\text{L}}{\text{A}}$ are next calculated for each member, and the corresponding quantities are all added together, thus enabling H to be calculated according to equation (4). In these summations due allowance must, of course, be made for the sign of the stress.

In the present case this calculation gives H = 16·7 tons. A new stress diagram (*d*) is then drawn from this value of H (*i.e.*, 1 – 4, 1′ – 4′ = 16·7 tons), and from this the stresses in the arched truss are found.

Reaction Locus for two-pinned Spandril Arch.—If the thrust and reactions be calculated for each node or panel point of the top flange, the reaction locus can be obtained as in the case of the arch ribs.

In order to save considerable labour, it is usual in the design of structures of this type to adopt a reaction locus given by some formula. The following loci are commonly given :—

(*a*) *Johnson's Parabolic Locus.*—This is of the form

$$h_1 = \frac{2{\cdot}5\,(r - d)\,x^2}{r\,\text{L}^2} + r + 2{\cdot}2\,d \quad \text{............(1)}$$

and was claimed by Professor J. B. Johnson to give results within 5 per cent.

(*b*) *Johnson's Hyperbolic Locus.*—This formula is a later one

by Professor Johnson, and is considered as giving results more nearly accurate than the parabolic locus. It is of the form

$$h_2 = \sqrt{\frac{8}{\cdot 3}\left(\frac{x}{l}\right)^2\left(Q^2 - K^2\right) + K^2}$$

where $Q = 1{\cdot}35\,r + 1{\cdot}85\,d$

$K = r + 2{\cdot}2\,d$

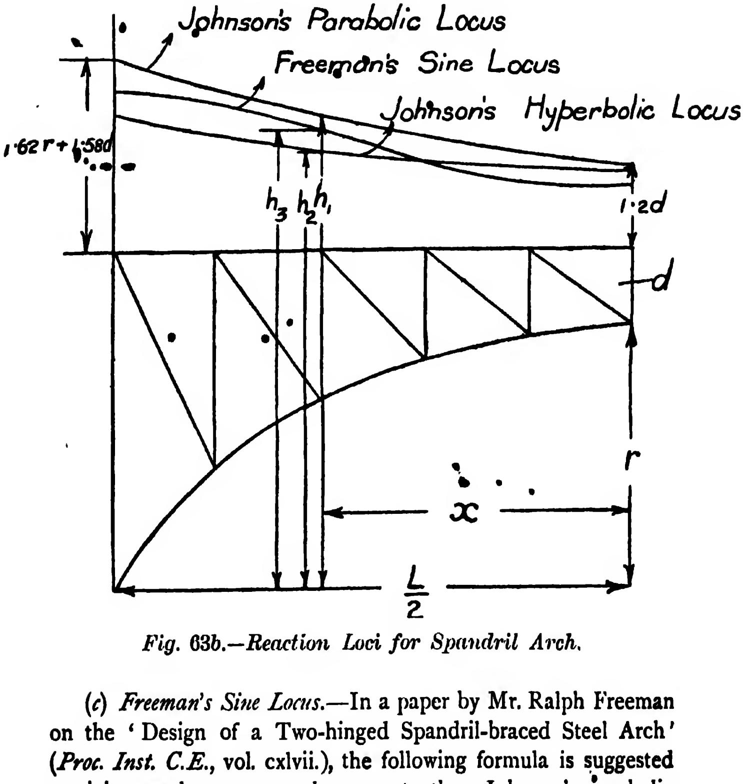


Fig. 63b.—Reaction Loci for Spandril Arch.

(*c*) *Freeman's Sine Locus.*—In a paper by Mr. Ralph Freeman on the 'Design of a Two-hinged Spandril-braced Steel Arch' (*Proc. Inst. C.E.*, vol. cxlvii.), the following formula is suggested as giving results more nearly accurate than Johnson's parabolic formula:—

$$h_3 = 1{\cdot}3\,(r + d) - {\cdot}2\,(r + d)\sin\left\{90^\circ - \frac{x}{\frac{1}{2}L}\cdot 180^\circ\right\}$$

In order to obtain an idea of the difference between these three formulæ, we will calculate their values for a number of values of x. We will take $L = 6r$ and $d = \frac{r}{5}$, the values of h being in terms of r.

x	h in terms of r.		
	Johnson's Parabolic Locus.	Johnson's Hyperbolic Locus.	Freeman's Sine Locus.
0	1·44	1·44	1·32
·1 L	1·46	1·45	1·37
·2 L	1·52	1·47	1·49
·3 L	1·62	1·51	1·63
·4 L	1·76	1·57	1·75
·5 L	1·94	1·63	1·80

The loci are shown in Fig. 63*b*.

The value of the reaction locus method in designing a spandril arch will at once be apparent, because no troublesome summations (needing, first of all, a knowledge of the dimensions of the various members) are necessary.

Numerical Example of Lengue Arch.—In order to give an example of the summation method of calculating the stresses in a two-pinned spandril arch, we will quote and consider the results for the Lengue arch, which was investigated by Mr. Freeman in the paper above referred to.

The arch carries the Benguella Railway over the River Lengue in Portuguese West Africa; the span is 138 ft., divided into ten panels; the depth at springing being 29 ft. 6 ins., and at crown 4 ft. 11 ins. (*i.e.*, rise = 24 ft. 5 in.); the width at top and bottom being 13 ft. 9 ins.

The stresses were first obtained by means of Johnson's parabolic locus, and from these stresses the values of the areas of

the various members were obtained. The stresses were then found by the method of summations. The method adopted in the summations was a variation upon that given in equation (4), page 148.

The stress F for an isolated load may be regarded as S × V, in the case where there is no load between the reaction and bay in which the member under consideration is situated, V being the vertical reaction at the point A or C. Take, for instance, the case of a load P at the node between the fourth and fifth bays, Fig. 64.

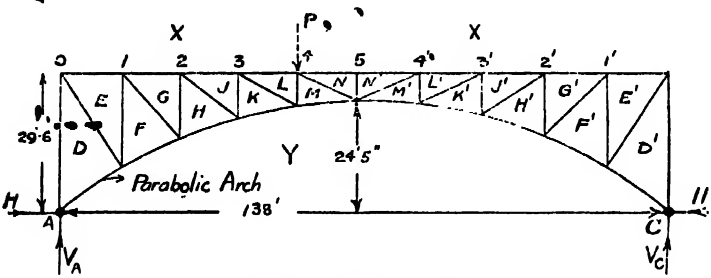


Fig. 64.—Lengue Arch.

S_{EF} represents the load or stress in any member, say E F, due to unit load at A, then V_A can be calculated easily, and $F_{EF} = V_A S_{EF}$.

Now suppose that our member is beyond the load, say the member E′ F′; it is clear that if we look at the structure from the other side, the stress in E′ F′ will be equal to $F_{E'F'} = V_C S_{EF}$

With this artifice for saving time in calculation our equation for H becomes

$$H = \frac{\Sigma \dfrac{V \cdot S U L}{A}}{\Sigma \dfrac{U^2 L}{A}}$$

The values of S and U can be calculated or else obtained by the reciprocal figures of Fig. 65, the upper diagram of which is for U and the lower for S. The results were then tabulated as shown in Table I.

From these figures Table II. is then constructed.

TABLE I.

	Member	L (feet)	A (sq.in.)	U	S	$\frac{U^2L}{A}$	$\Sigma\frac{U^2L}{A}$	$\frac{SUL}{A}$	$\Sigma\frac{SUL}{A}$
Top Chord.	E X	13·8	15·3	0·43	0·6	0·2	—	0·3	0·3
	G X	,,	15·3	1·15	2·0	1·2	—	2·1	2·4
	J X	,,	15·3	2·36	4·7	5·0	—	10·0	12·4
	L X	,,	21·2	4·04	9·2	10·6	—	24·2	36·6
	N X	,,	21·2	5·00	14·0	16·4	33·4 × 2	45·6	82·2
	N′ X	,,	21·2	5·00	14·0	—		45·6	127·8
	L′ X	,,	21·2	4·04	14·0	—	= 66·8	36·9	164·7
	J′ X	,,	15·3	2·36	10·9	—	—	23·2	187·9
	G′ X	,,	15·3	1·15	6·0	—	- -	8·3	196·2
	E′ X	,,	15·3	0·43	6·0	—	—	2·4	198·6
Bottom Chord.	D Y	16·4	30·4	1·20	0·0	0·8	—	0·0	0·0
	F Y	15·4	22·9	1·60	0·7	1·7	—	0·7	0·7
	H Y	14·6	22·9	2·30	2·1	3·4	—	3·1	3·8
	K Y	14·2	22·9	3·42	4·8	7·2	—	10·6	14·4
	M Y	13·9	21·2	5·06	9·3	16·7	29·8 × 2	30·9	45·3
	M′ Y	13·9	21·2	5·06	14·0	—		46·5	91·8
	K′ Y	14·2	22·9	3·42	11·2	—	= 59·6	23·8	115·6
	H′ Y	14·6	22·9	2·30	8·5	—	—	12·4	128·0
	F′ Y	15·4	22·9	1·00	6·8	—	—	7·3	135·3
	D′ Y	16·4	30·4	1·20	5·6	—	—	3·6	138·9
Verticals.	X D	29·5	15·3	0·65	1·0	0·8	—	1·3	1·3
	E F	20·6	15·2	0·72	1·3	0·7	—	1·3	2·6
	G H	13·8	14·5	0·77	1·7	0·6	—	1·2	3·8
	J K	8·9	13·7	0·72	2·0	0·3	2·5 × 2	1·9	4·7
	L M	5·9	12·9	0·36	1·6	0·1		0·3	5·0
	N N′	5·1	—	0·0	0·0	0·0	= 5·0	0·0	5·0
	M′ L′	5·9	12·9	0·36	0·0	—	—	0·0	5·0
	K′ J′	8·9	13·7	0·72	1·3	—	—	0·6	5·6
	H′ G′	13·8	14·5	0·77	1·8	—	—	1·3	6·9
	F′ E′	20·6	15·2	0·72	2·0	—	—	2·0	8·9
	D′ X	29·5	15·3	0·65	2·0	—	—	2·5	11·4
Diagonals.	D E	24·8	10·3	0·78	1·2	1·5	—	2·2	2·2
	F G	19·5	10·3	1·02	1·9	2·0	—	3·7	5·9
	H J	16·3	10·3	1·43	3·2	3·2	—	7·3	13·2
	K L	15·0	11·4	1·82	5·0	4·3	—	12·0	25·2
	M N	14·8	21·2	1·02	5·0	0·7	11·7 × 2	3·6	28·8
	N′ M′	14·8	21·2	1·02	0·0	—	= 23·4	0·0	28·8
	L′ K′	15·0	11·4	1·83	3·3	—	—	7·8	36·6
	J′ H′	16·3	10.3	1·43	3·4	—	—	7·7	44·3
	G′ F′	19·5	10·3	1·02	2·8	—	—	5·4	49·7
	E′ D′	24·8	10·3	0·78	2·4	—	—	4·5	54·2

Total $\Sigma\frac{U^2L}{A} = 154·8$

TABLE II.

THRUSTS DUE TO UNIT LOADS AT JOINTS.

Position of Load	Reactions	Values of $\frac{\Sigma SUL}{A}$: Top Chord	Bottom Chord	Verticals	Diagonals	Total	$V\Sigma\frac{SUL}{A} = \Sigma\frac{FUL}{A}$		$H = \frac{\Sigma\frac{FUL}{A}}{\Sigma\frac{U^2L}{A}}$	H by Johnson's Locus	H by Freeman's Locus
Joint 1	$V_A = \cdot 9$	0·3	0	2·6	2·2	5·1	4·5	43·3	0·28	0·29	0·28
	$V_C = \cdot 1$	196·2	135·3	6·9	49·7	388·1	38·8				
Joint 2	$V_C = \cdot 8$	2·4	0·7	3·8	5·9	12·8	10·2	83·4	0·54	0·56	0·54
	$V_A = \cdot 2$	187·9	128·0	5·6	44·3	365·8	73·2				
Joint 3	$V_C = \cdot 7$	12·4	3·8	4·7	13·2	34·1	23·8	120·4	0·78	0·77	0·78
	$V_A = \cdot 3$	164·7	115·6	5·0	36·6	321·9	96·6				
Joint 4	$V_C = \cdot 6$	36·6	14·4	5·0	25·2	81·2	48·7	149·9	0·97	0·92	0·98
	$V_A = \cdot 4$	127·8	91·8	5·0	28·8	253·4	101·2				
Joint 5	$V_C = \cdot 5$	82·2	45·3	5·0	28·8	161·3	80·6	161·3	1·04	0·97	1·06
	$V_A = \cdot 5$	82·2	45·3	5·0	28·8	161·3	80·6				

In order to make Table II. quite clear, we will give a little additional explanation of the manner in which it is compiled. Take, for instance, the case where the load is at joint 2: there are two top chord members to the left, viz., E X and G X, and $\frac{\Sigma S U L}{A}$ for these = 2·4 from Table I.; to the right there are eight top chord members, and $\frac{\Sigma S U L}{A}$ for these will be, from Table I., 187·9, because this is the same as the point 2′ working

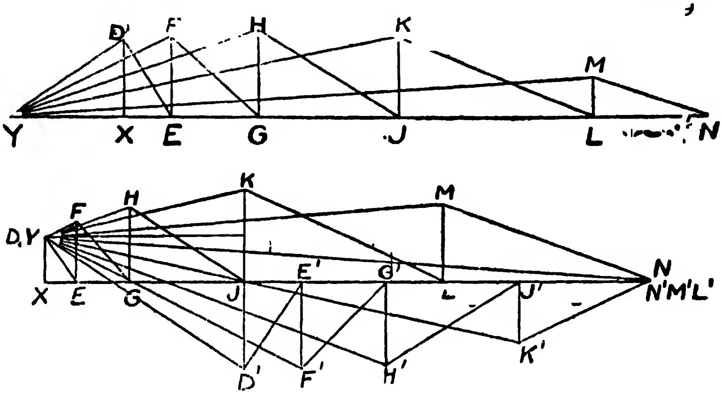


Fig. 65.—Reciprocal Diagrams for S and U.

from the other end. The values of the other members are obtained in a similar manner. The column of $\frac{\Sigma F U L}{A}$ is obtained by multiplying 12·8 by ·8, giving 10·2, and 365·8 by ·2, giving 73·2, and adding these results together; these values are then divided by 154·8, which is the value of $\frac{\Sigma U^2 L}{A}$ for all the members, to give the value of A.

Having thus calculated the thrusts, the stresses in the various members are obtained by reciprocal diagrams; those for loads at joints 4 and 5 being shown in Fig. 66, and from these Table III. is compiled. From these figures the maximum stresses in the members can be obtained for a given load system, and these are combined with the dead load, temperature, and wind stresses to give the final figures for design.

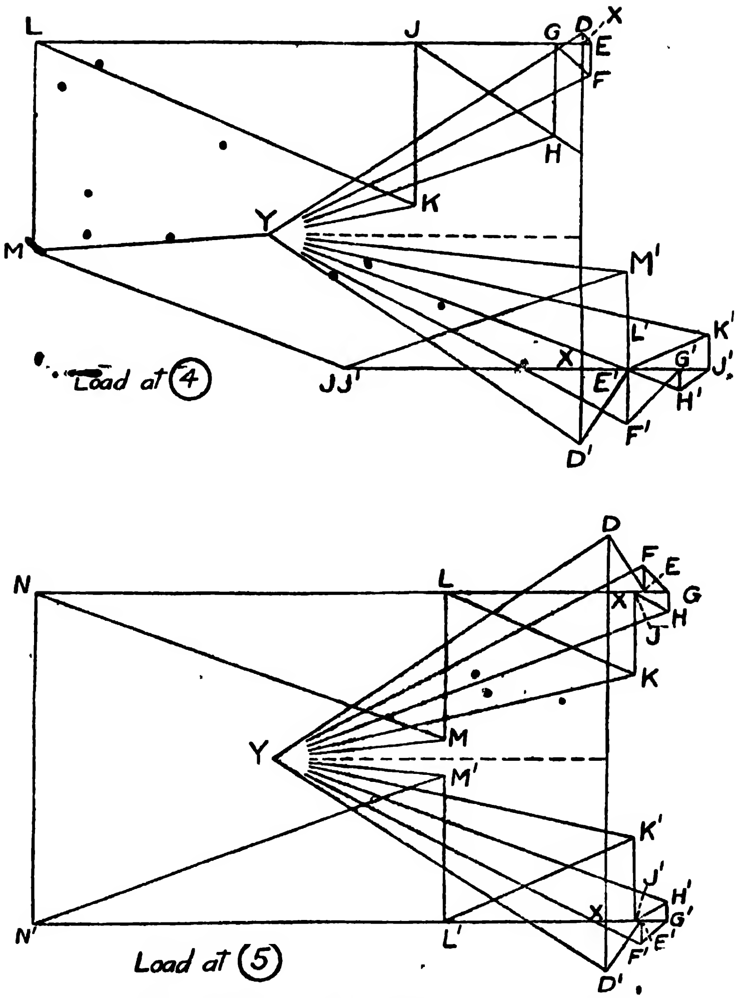


Fig. 66.—Reciprocal Diagrams for Lengue Arch.

TABLE III.

Stresses due to Unit Load at each Joint.

Position of Load Joint	Member									
	EX	GX	JX	LX	NX	DY	EY	HY	KY	MY
1	+0·48	+0·47	+0·42	+0·26	−0·02	+0·33	−0·22	−0·20	−0·14	+0·02
2	+0·30	+0·98	+0·91	+0·63	+0·10	+0·64	+0·27	−0·47	−0·38	−0·09
3	+0·13	+0·51	+1·45	+1·08	+0·30	+0·93	+0·73	+0·29	−0·68	−0·29
4	−0·02	+0·09	+0·53	+1·72	+0·75	+1·16	+1·11	+0·94	+0·46	−0·74
5	−0·11	−0·19	−0·11	+0·51	+1·79	+1·24	+1·20	+1·30	+1·16	+0·54
6	−0·15	−0·32	−0·40	−0·15	+0·75	+1·16	+1·26	+1·37	+1·41	+1·13
7	−0·13	−0·29	−0·41	−0·30	+0·30	+0·93	+1·03	+1·14	+1·22	+1·09
8	−0·10	−0·21	−0·32	−0·28	+0·10	+0·64	+0·71	+0·80	+0·88	+0·83
9	−0·05	−0·12	−0·18	−0·18	−0·02	+0·33	+0·37	+0·42	+0·47	+0·47
Sum +	0·91	2·05	3·31	4·20	4·09	7·36	6·68	6·26	5·60	4·08
Sum −	0·56	1·13	1·42	0·91	0·04	—	0·22	0·67	1·20	1·12

Position of Load Joint	Member									
	DX	EF	GH	JK	LM	DE	FG	HJ	KL	MN
0	+1·00	—	—	—	—	—	—	—	—	—
1	+0·72	+1·00	−0·03	−0·08	−0·10	−0·86	+0·01	+0·06	+0·18	+0·29
2	+0·45	+0·68	+0·96	−0·12	−0·19	−0·54	−0·96	+0·08	+0·30	+0·57
3	+0·20	+0·38	+0·60	+0·85	−0·28	−0·24	−0·53	−1·12	+0·41	+0·82
4	−0·03	−0·10	+0·29	+0·50	+0·65	+0·03	−0·15	−0·52	−1·29	+1·03
5	−0·17	−0·08	+0·07	+0·26	+0·46	+0·20	+0·11	−0·11	−0·66	−1·36
6	−0·23	−0·16	−0·06	+0·11	+0·32	+0·27	+0·23	+0·10	−0·28	−0·95
7	−0·20	−0·16	−0·08	+0·05	+0·22	+0·24	+0·22	+0·15	−0·12	−0·64
8	−0·15	−0·12	−0·07	+0·02	+0·14	+0·18	+0·17	+0·12	−0·04	−0·40
9	−0·08	−0·07	−0·07	—	+0·06	+0·09	+0·09	+0·07	—	−0·18
Sum +	2·37	2·06	1·92	1·79	1·85	1·01	0·83	0·58	0·89	2·71
Sum −	0·86	0·69	0·31	0·20	0·57	1·64	1·64	1·75	2·39	3·53

Stress in N N′ due to load at 5 = 1·00.

A load at joint 0 produces a small thrust, but the stresses in all the members except D X are so small that they may be neglected.

FIXED OR HINGELESS ARCH RIBS.

In the case of an arch rib in which the ends have no hinges but are firmly fixed in direction, we have the following conditions to satisfy.

(*a*) Horizontal movement of end C = 0

(*b*) Vertical movement of end C = 0

(*c*) Change of slope of end C = 0

These give rise to the following equations:—

(*a*) gives from equation 3*a*, p. 122

$$\int_A^C \frac{M\,y}{E\,I}\,d\,s = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1a)$$

or, avoiding the calculus, if the ordinates are measured at equal short finite distances $\delta\,s$ apart,

$$\sum_A^C \frac{M\,y}{E\,I} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1b)$$

(*b*) gives from equation 3*b*, p. 122

$$\int_A^C \frac{M\,x}{E\,I}\,d\,s = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

or for finite elements

$$\sum_A^C \frac{M\,x}{E\,I} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2b)$$

(*c*) Since the change of curvature of a beam is equal to $\frac{M}{E\,I}$, and the change of curvature of a short arc $\delta\,s$ is equal to $\frac{\delta\,\theta}{\delta\,s}$, where $\delta\,\theta$ is the angle turned through by the end, $\delta\,\theta = \frac{M\,\delta\,s}{E\,I}$, and the total change of slope at any point will be equal to $\Sigma\,\frac{M\,\delta\,s}{E\,I}$.

Since this must be zero at C we get

$$\sum_A^C \frac{M}{E\,I} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3a)$$

or in the notation of the calculus

$$\int_{A}^{C} \frac{M\,ds}{EI} = 0 \quad \text{.........................} (3b)$$

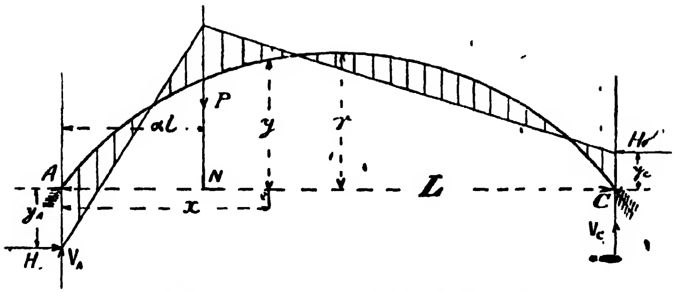


Fig. 67.—Fixed or Hingeless Arch Ribs.

Parabolic Arch Rib with Isolated Load.—If, as in the case of the parabolic two-pinned arch, the section varies so that $\frac{\delta s}{I} = \frac{\delta x}{I_0}$ we shall get the following results for an isolated load P, Fig. 67, end bending moments being induced in a manner similar to that which occurs with ordinary beams with ends fixed or built-in.

We now get the following equations :—

$$V_A + V_C = P \quad \text{.........................} (4)$$

By taking moments about the point A we shall get

$$P\,a\,L - V_C\,L - H\,y_C + H\,y_A = 0 \quad \text{...........} (5)$$

and equations (1*a*) to (3*a*) become

$$\int_{0}^{L} M\,y\,dx = 0 \quad \text{.........................} (6)$$

$$\int_{0}^{L} M\,x\,dx = 0 \quad \text{.........................} (7)$$

$$\int_{0}^{L} M\,dx = 0 \quad \text{.........................} (8)$$

As we shall show later, the following results can be obtained from these equations :—

$$V_A = P(1 - a)^2 (1 + 2a) \quad \ldots\ldots (9)$$

$$V_C = P(3 - 2a)\,a^2 \quad \ldots\ldots (10)$$

$$H = \frac{15\,P\,L}{4\,r}(1 - a)^2\,a^2 \quad \ldots\ldots (11)$$

$$y_A = \frac{(10\,a - 4)\,r}{15\,a} \quad \ldots\ldots (12)$$

$$\bar{y}_C = \frac{(6 - 10\,a)\,r}{15\,(1 - a)} \quad \ldots\ldots (13)$$

These results can be tabulated in convenient form as follows :—

a	$\frac{V}{P}$	$\frac{V_C}{P}$	H (in terms of $\frac{PL}{r}$)	$\frac{y_A}{r}$	$\frac{y}{r}$	
·00	1·000	·000	·000	− ∞	+ ·400	1·00
·05	·993	·007	·008	− 4·667	+ ·386	·95
·10	·972	·028	·030	− 2·000	+ ·370	·90
·20	·896	·104	·096	− ·667	+ ·333	·80
·25	·844	·156	·132	− ·400	+ ·311	·75
·30	·784	·216	·165	− ·222	+ ·286	·70
·40	·648	·352	·216	− ·000	+ ·222	·60
·50	·500	·500	·234	+ ·133	+ ·133	·50
	$\frac{V_C}{P}$	$\frac{V_A}{P}$	H	$\frac{y_C}{R}$	$\frac{y_A}{r}$	a

Fig. 67 is drawn for the case of $a = {\cdot}25$.

Proof of Formulæ.—As in the case of the two-pinned arch we will give the proof of the formulæ.

Take first equation—(8):

From A to N, Fig. 67, $M = V_A . x - H(y - y_A)$

N to C, $M = V_A x - H(y - y_A) - P(x - \alpha L)$

$$\therefore \int^L M\,dx = \int^L V_A . x\,dx - \int^L H(y - y_A)\,dx$$

$$- \int^L P(x - \alpha L)\,dx$$

As before $y = \frac{4r}{L^2}(Lx - x^2)$

$$\text{Now} \int^L V_A\,x\,dx = \frac{V_A L^2}{2}$$

$$\int_0^L H(y - y_A)\,dx = -H y_A L + \frac{4rH}{L^2}\left(\frac{L^3}{2} - \frac{L^3}{3}\right)$$

$$\int_{\alpha L}^{L} P(x - \alpha L)\,dx = \left[P\left(\frac{x^2}{2} - \alpha L x\right)\right]_{\alpha L}^{L}$$

$$= P\left(\frac{L^2}{2} - \alpha L^2 - \frac{\alpha^2 L^2}{2} + \alpha^2 L^2\right)$$

$$= \frac{PL^2}{2}(1 - 2\alpha + \alpha^2) = \frac{PL^2}{2}(1 - \alpha)^2$$

$$\therefore \int_0^L M\,dx = \frac{V_A L^2}{2} - HL\left(\frac{4r}{6} - y_A\right) - \frac{PL^2}{2}\left(1 - \alpha\right)^2$$

$\therefore$ Dividing through by $\frac{L}{2}$ equation (8) becomes

$$V_A L - 2H\left(\frac{2r}{3} - y_A\right) - PL(1 - \alpha)^2 = 0 \quad \text{...... (14)}$$

Next take equation (7): The values of M are as before, $\therefore$ we have

$$\int_0^L M\,x\,dx = \int_0^L V_A\,x^2\,dx - \int_0^L H\,(y - y_A)\,x\,dx$$

$$- \int_{aL}^L P\,(x - a\,L)\,x\,dx$$

$$\int_0^L V_A\,x^2\,dx = \frac{V_A\,L^3}{3}$$

$$\int_0^L H\,(y - y_A)\,x\,dx = \frac{4\,r\,H}{L^2}\left(\frac{L^4}{3} - \frac{L^4}{4}\right) - H\,y_A\,\frac{L^2}{2}$$

$$\int_{aL}^L P\,(x - a\,L)\,x\,dx = \left[P\left(\frac{x^3}{3} - \frac{a\,L\,x^2}{2}\right)\right]_{aL}^{L}$$

$$= P\left(\frac{L^3}{3} - \frac{a\,L^3}{2} - \frac{a^3\,L^3}{3} + \frac{a^3 L^3}{2}\right)$$

$$= \frac{P\,L^3}{6}\left(2 - 3a + a^3\right)$$

$$= \frac{P\,L^3}{6}\,(1 - a)^2\,(2 + a)$$

$$\therefore \int_0^L M\,x\,dx = \frac{V_A\,L^3}{3} - H\left(\frac{r\,L^2}{3} - \frac{y_A\,L^2}{2}\right)$$

$$- \frac{P\,L^3}{6}\,(1 - a)^2\,(2 + a)$$

$\therefore$ Equation (7) becomes, dividing through by $\dfrac{L^2}{3}$

$$V_A\,L - H\left(r - \frac{3\,y_A}{2}\right) - \frac{P\,L}{2}\,(1 - a)^2\,(2 + a) = 0 \quad \ldots \quad (15)$$

Finally take equation (6), using the same values of M:

$$\int_0^L M\,y\,dx = \int_0^L V_A\,x\,y\,dx - \int_0^L H\,(y - y_A)\,y\,dx$$

$$- \int P\,(x - a\,L)\,y\,dx$$

$$\int_0^L V_A\, x\, y\, d x = \int_0^L V_A \frac{4 r}{L^2}(L x - x^2)\, x\, d x$$

$$\frac{V_A \cdot 4 r}{L^2}\left(\frac{L^4}{3} - \frac{L^4}{4}\right)$$

$$\frac{4 V_A\, r\, L^2}{12} = \frac{V_A\, r\, L^3}{3}$$

$$\int_0^L H (y - y_A)\, y\, d x = \int_0^L H \cdot y^2\, d x - \int_0^L H\, y_A \cdot y\, d x$$

$$\text{1st term} = H \int_0^L \frac{16 r^2}{L^4}(L^2 x^2 - 2 L x^3 + x^4)\, d x$$

$$= \frac{16 r^2 H}{L^4}\left(\frac{L^5}{3} - \frac{2 L^5}{4} + \frac{L^5}{5}\right)$$

$$= \frac{8 r^2 H L}{15}$$

$$\text{2nd term} = \int_0^L H\, y_A \frac{4 r}{L^2}(L x - x^2)\, d x$$

$$= \frac{H\, y_A\, 4 r}{L^2}\left(\frac{L^3}{2} - \frac{L^3}{3}\right)$$

$$= \frac{2 r H y_A L}{3}$$

$$\int_{a L}^L P (x - a L)\, y\, d x = \int_{a L}^L P\, x\, y\, d x - \int_{a L}^L P\, a\, L\, y\, d x$$

$$= P \int_{a L}^L \frac{4 r}{L^2}(L x - x^2)\, x\, d x - P\, a\, L \int_a^L \frac{4 r}{L^2}(L x - x^2)\, d x$$

$$= 4 \frac{P r}{L^2}\left[\frac{L x^3}{3} - \frac{x^4}{4}\right]_{a L}^{L} - \frac{P\, a\, 4 r L}{L^2}\left[\frac{L x^2}{2} - \frac{x^3}{3}\right]_{a L}^{L}$$

$$= 4 \frac{P r}{L^2}\left\{\frac{L^4}{3} - \frac{L^4}{4} - \frac{a^3 L^4}{3} - \frac{a^4 L^4}{4}\right\}$$

$$- 4 \frac{P r \cdot a L}{L^2}\left\{\frac{L^3}{2} - \frac{L^3}{3} - \frac{a^2 L^3}{2} + \frac{a^3 L^3}{3}\right\}$$

$$= 4\,P\,r\,L^2\left\{\frac{1}{12} - \frac{a^3}{3} + \frac{a^4}{4} - \frac{a}{6} + \frac{a^3}{2} - \frac{a^4}{2}\right\}$$

$$= 4\,\frac{P\,r\,L^2}{12}\left\{1 - 2a + 2\,a^3 - a^4\right\}$$

$$= \frac{P\,r\,L^2}{3}(1 - a)^3(1 + a)$$

∴ Collecting together, we get:

$$\int_0^L M\,y\,d\,x = \frac{V_A\,.\,r\,L^2}{3} - \frac{8\,r^2\,H\,L}{15} + \frac{2\,r\,H\,y_A\,L}{3}$$

$$- \frac{P\,r\,L^2}{3}(1 - a)^3(1 + a)$$

Dividing through by $\frac{r\,L}{3}$, equation (6) becomes:

$$V_A\,L - H\left(\frac{8\,r}{5} - 2\,y_A\right)$$

$$- P\,L\,(1 - a)^3(1 + a) = 0 \quad \ldots\ldots\ldots\ldots(16)$$

Subtract equation (15) from (14) and we get:

$$H\left(\frac{r}{3} - \frac{y_A}{2}\right) = +\frac{a}{2}\,P\,L\,(1 - a)^2 \quad \ldots\ldots\ldots\ldots\ldots(17)$$

Subtract equation (16) from (15), thus getting:

\ 5 2 / 2

Divide (17) by (18), then:

$$\frac{\frac{r}{3} - \frac{y_A}{2}}{\frac{3\,r}{5} - \frac{y_A}{2}} = \frac{1}{2\,a + 1}$$

$$\therefore\ -\frac{y_A}{2}(2\,a + 1 - 1) = \frac{3\,r}{5} - \frac{r}{3}(2\,a + 1)$$

$$-\,a\,y_A = \frac{9\,r - 10\,a\,r - 5\,r}{15}$$

$$y_A = \frac{(10\,a - 4)\,r}{15\,a} \text{ as in equation (12)}$$

Now put this value in equation (17), then

$$H\left\{\frac{r}{3} - \frac{(5\alpha - 2)r}{15\alpha}\right\} = \frac{\alpha}{2} P L (1 - \alpha)^2$$

$$\therefore H r\left\{\frac{5\alpha - 5\alpha + 2}{15\alpha}\right\} = \frac{2 H r}{15\alpha} = \frac{\alpha}{2} P L (1 - \alpha)^2$$

$$\therefore H = \frac{15(1 - \alpha)^2 \alpha^2 P L}{4r} \text{ as in equation (11)}$$

Put these results in equation (16) and we get:

$$V_A L = \frac{15(1 - \alpha)^2 \alpha^2 P L}{2r}\left(\frac{2r}{3} - \frac{(10\alpha - 4)r}{15\alpha}\right) + P L (1 - \alpha)^2$$

$$= P L (1 - \alpha)^2 \left\{\frac{15\alpha^2}{2r}\left(\frac{4r}{15\alpha}\right) + 1\right\}$$

$$= P L (1 - \alpha)^2 (1 + 2\alpha)$$

$$V_A = P(1 - \alpha)^2 (1 + 2\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\therefore V_C = P - V_A = P(3 - 2\alpha)\alpha^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Finally a substitution in equation (5) gives:

$$y_C = \frac{(6 - 10\alpha)r}{15(1 - \alpha)}$$

Reaction Locus for Hingeless Parabolic Arch Rib.—We can obtain the reactions in the present case of the hingeless parabolic arch by a similar construction to that which we used for the arch with the two hinges; in the present case, however, we require two curves instead of one.

The intersection of the reactions will be along a straight line of height $h = 1{\cdot}2\,r$ (Fig. 68) and the reactions will be tangential to the line M N Q. This can be seen from Fig. 67, from which we get:

$$\frac{h - y_A}{\alpha L} = \frac{V_A}{H} = \frac{P(1 - \alpha)^2(1 + 2\alpha)}{\frac{15 P L}{4r}(1 - \alpha)^2 \alpha^2} = \frac{4r(1 + 2\alpha)}{15 L \alpha^2}$$

$$\therefore h - y = \frac{4r(1 + 2\alpha)}{15\alpha}$$

$$\therefore \quad h = \frac{4r(1 + 2a)}{15a} + y_A$$

$$= \frac{4r(1 + 2a) + (10a - 4)r}{15a}$$

$$= \frac{r(4 + 8a + 10a - 4)}{15a}$$

$$= \frac{18ar}{15a} = 1 \cdot 2r$$

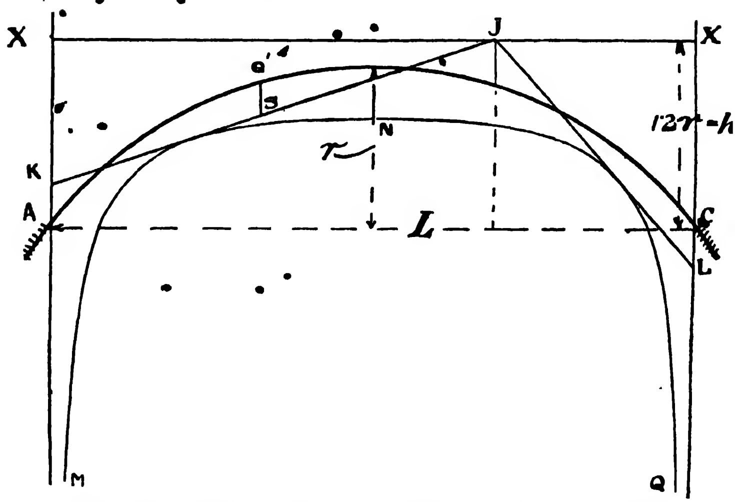


Fig. 68.—Reaction Loci for Hingeless Parabolic Arch.

The line M N Q can be drawn by taking from the table on p. 161 the values of y_A and y_C for values of a from 0 to ·5, then by drawing in the reactions at each point we shall be able to draw in the required curve, which is the 'envelope' of the reactions. This curve is sometimes taken as two hyperbolas, the height at the centre being $\frac{2r}{3}$.

* **Hingeless Circular Arch Rib.**—In dealing with the hingeless circular arch rib, the equations become too trouble-

some for simplification into general formulæ. Formulæ will be found in Green's *Theory of Arches*, which can be solved when particular values for the angles are inserted. In the case of a semicircular arch rib with fixed ends and a load at the crown, the formulæ become simple and can be deduced as follows :—

The equations to be satisfied are

$$\int_A^C M\,d s = 0 \qquad \text{(1)}$$

$$\int_A^C M\,y\,d s = 0 \qquad \text{(2)}$$

$$\int_A^C M\,x\,d s = 0 \qquad \text{(3)}$$

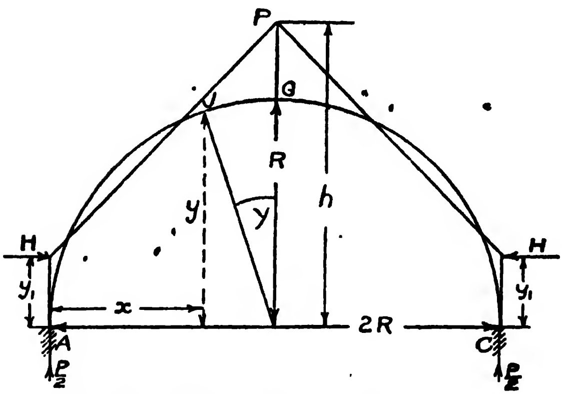


Fig. 69.—Hingeless Semicircular Arch.

Considering a point J, Fig. 69, we see that from symmetry $y_A = y_C = y_1$; $V_A = V = \frac{P}{2}$; and that

$$\frac{H}{V_A} = \frac{R}{(h - y_1)}$$

$$\therefore\ H = \frac{P\,R}{2\,(h - y_1)} \qquad \text{(4)}$$

Bending moment at J $= M_J = \frac{P\,x}{2} - H\,(y - y_1)$

$$= \frac{P\,R}{2}(1 - \sin\gamma) - \frac{P\,R}{2\,(h - y_1)}(R\cos\gamma - y_1)$$

$$= \frac{P\,R}{2}\left\{(1 - \sin\gamma) - \left(\frac{R\cos\gamma - y_1}{h - y_1}\right)\right\} \quad \dots\dots\dots \ (5)$$

From (1) $\qquad \int_A^C M\,ds = 0 = R\int_A^C M\,d\gamma$

and since from symmetry $\int_C^P M\,ds = \int_P^A M\,ds$, we have

$$\int_{\frac{\pi}{2}}^{0} M\,d\gamma = 0$$

$$\int_{\pi}^{0} \frac{P\,R}{2}\left\{(1 - \sin\gamma) - \frac{(R\cos\gamma - y_1)}{h - y_1}\right\}d\gamma = 0$$

i.e. $\left[\gamma + \cos\gamma - \frac{R\sin\gamma - y_1\gamma}{h - y_1}\right]_{\pi}^{0} \quad 0$

i.e. $\left[\frac{\pi}{2} + 0 - \left(\frac{R - y_1\frac{\pi}{2}}{h - y_1}\right) - 1\right] = 0$

i.e. $(h - y_1)\left(\frac{\pi}{2} - 1\right) - R + \frac{y_1\pi}{2} = 0$

$$\left(\frac{\pi}{2} - 1\right) + y_1 = R$$

$$h\,(\pi - 2) + 2\,y_1 = 2\,R \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \ (6)$$

Similarly from equation (2) $\int^{0} M\,y\,d\gamma = 0$

i.e. $\int^{0} \frac{P\,R}{2}\left\{1 - \sin\gamma) - \frac{(R\cos\gamma - y_1)}{h - y_1}\right\} R\cos\gamma\,d\gamma$

$$i.e. \int_{\frac{\pi}{2}}^{0} \left\{ \cos\gamma - \cos\gamma\sin\gamma - \frac{(R\cos^2\gamma - y_1\cos\gamma)}{(h - y_1)} \right\} d\gamma = 0$$

$$i.e. \left[\sin\gamma + \frac{\cos 2\gamma}{4} - \frac{\left\{ R\left(\frac{\gamma}{2} + \frac{\sin 2\gamma}{4}\right) - y_1 \sin\gamma \right\}}{h - y_1} \right]_{\frac{\pi}{2}}^{0} = 0$$

$$i.e. \left[1 - \frac{1}{4} - \frac{\left\{ R\left(\frac{\pi}{4} + 0\right) - y_1 \right.}{h - y_1} - \frac{1}{4} \right] = 0.$$

$$i.e. \left[\frac{1}{2}(h - y_1) - \frac{\pi R}{4} + y_1 \right] = 0$$

$$\frac{h}{2} + \frac{y_1}{2} = \frac{\pi R}{4}$$

$$2h + 2y_1 = \pi R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Since we have only two unknowns, h and y_1, we need not consider equation (3), because from symmetry $y_1 = y_2$.

From equation (6) and (7) we get

$$h = R \cdot \frac{\pi - 2}{4 - \pi} = 1{\cdot}33\,R \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$y_1 = \frac{R \cdot (4 + 2\pi - \pi^2)}{2(4 - \pi)} = {\cdot}24\,R \quad \ldots\ldots\ldots \quad (9)$$

Deflections in Arch Ribs.—We have seen that the vertical deflection of an arch rib due to bending is given by the formula

$$\delta_1 = \int \frac{M\,x\,ds}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$= \int \frac{M\,x\,dx}{E\,I_0} \quad \text{(approx.)} \quad \ldots\ldots\ldots\ldots(2)$$

Now the deflection at each end is zero, so imagine that the span is divided into a number of segments and that the bending moment M at each section is treated as a load on a beam of the same span of the arch, then if M be considered as an imaginary load, the ordinate of the B.M. diagram drawn with a polar distance equal to $E\,I_0$ will give the deflections at each point.

Thus the deflections in the arch due to bending will be the same as in a beam of the same span and section of the arch subjected to the same bending moments as the arch.

Deflection due to Thrust.—In addition to the deflection due to bending there will be a deflection due to the deformation caused by the direct thrust.

This thrust deflection at any point whose ordinate is y is approximately given by

$$\delta_2 = \frac{2\,H\,y}{E\,A} \qquad (3)$$

Comparative Results for two-pinned three-pinned, and built-in Arches.—In a very thorough investigation of the stresses of arches* Mr. Atcherley and Prof. Karl Pearson have made comparative calculations for a circular arch of 108 metres span and 6·5 metres rise, these being the approximate dimensions of the Pont Alexandre III. in Paris.

The assumed loading was a dead load of 1 ton per metre run over the whole span and a live load of $\frac{1}{2}$ ton per metre run from one abutment to the centre. The assumed section was of **I** form with A = 63, I = 12,660, depth = 31, all in inch units.

The following is a summary of the results obtained :—

	Two Hinges.	Three Hinges.	No Hinges.
Horizontal thrust (tons)	277	280	272
Temperature thrust for 40° C. variation (tons)	2·42		14
Maximum stress (tons per sq. in.), excluding temperature ...	9·28	9·31	10·76 (terminal) 7·54 (intermediate)
Maximum deflection in cms.	34	25·6	30

* *The Graphics of Metal Arches* (Drapers' Company Research Memoirs, Dulau & Co., London).

CHAPTER IX.

STRESSES IN PORTALS AND WIND BRACINGS.

PORTALS.

PORTALS of various forms are found very commonly in bridge-work and in steel skeleton and reinforced concrete constructions for buildings, and are used most commonly to resist horizontal forces, principally those caused by wind.

A portal consists of two parallel members A E, B F, Fig. 70,

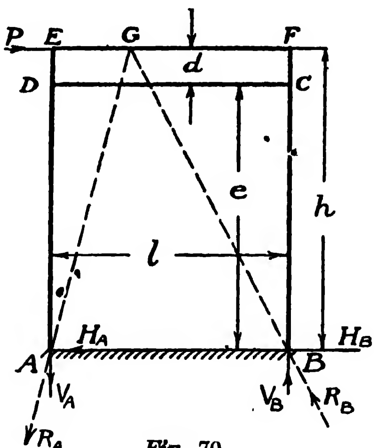


Fig. 70.

which are usually vertical and form the columns in buildings, connected at one end by a member E D F C, which may be a solid girder or may be of framed form as in Figs. 79 and 80.

The stresses in such a structure are not statically determinate, and are affected by the relative stiffness of the members composing it and by the manner in which the ends of the columns are fixed.

We know that the reactions R_A and R_B must intersect at a point G upon the line of action of the resultant horizontal force P, which is commonly the only force considered. We can also establish the following equations by taking moments about A and B and equating the horizontal components of R_A and R_B to the force P.

$$V_B = \frac{P h}{l} \quad \dots\dots\dots \quad (1)$$

$$V_A = \frac{-P h}{l} = -V_B \quad \dots\dots\dots \quad (2)$$

$$H_A + H_B = P \quad \dots\dots\dots \quad (3)$$

The relative proportions of H_A and H_B depend upon the relative stiffnesses of the two columns and the compressibility and rigidity of the cross-beam as well as upon the methods of fixing.

SOLID GIRDER PORTALS.

We will take various cases that may arise.

Cross-beam Stiff and Incompressible and rigidly connected to the Columns.

(a) Columns Pin-jointed at Ends.—1. *Columns of same Length and Stiffness; Single Force P at Top.*—In this case, Fig. 71, the columns are pin-jointed at A, and as they are exactly similar each column bends in exactly the same manner; since, therefore, the cross-beam is incompressible, it follows that each column must deflect by the same amount, and therefore that

$$H_B = \frac{P}{2} = H_A \quad \dots\dots\dots \quad (4)$$

The columns act as cantilevers loaded at their ends, and so their B.M., Shear, and Thrust diagrams for the right-hand are as shown in the diagrams; those for the left-hand column will be the same, except that the thrust will be negative. Considering the cross-beam, we see that as the connection with the columns is rigid, bending moments will be induced in it as shown in the figure; *e* and *h* being nearly equal.

(2) *Columns of same Lengths and Stiffness; uniformly distributed Load on one Column.*—Fig. 72 shows the forces acting in this case. Since the cross-beam is rigid, we may regard the columns as cantilevers; the column B C carries a load H_B at the free end,

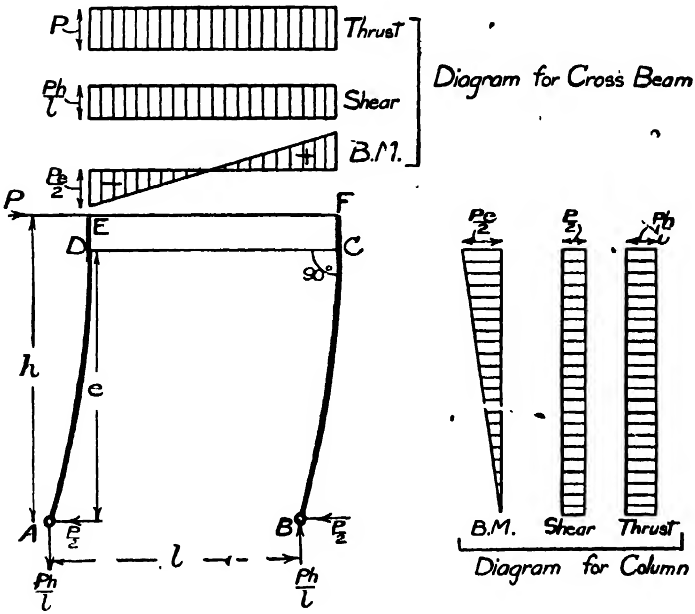


Fig. 71.

and the column A D carries a load H_A at the end and a uniformly distributed load in a direction reverse to H_A; these cantilevers must have deflections δ_D, δ_C which are equal since the cross-beam is incompressible. From the usual formulæ for deflections of cantilevers we get

$$\delta_C = \frac{H_B \cdot e^3}{3\,E\,I} \quad \text{..............................} \quad (5)$$

$$\delta_D = \frac{H_A \cdot e^3}{3\,E\,I} - \frac{P\,e^3}{8\,E\,I} \quad \text{..................} \quad (6)$$

I is the same in each case, because we have assumed the columns to be of the same stiffness.

Since $\delta_C = \delta_D$

$$\frac{H_B}{3} = \frac{H_A}{3} - \frac{P}{8} \quad \text{.........................} \quad (7)$$

$$\text{or } H_A - H_B = \frac{3P}{8} \quad \text{.........................} \quad (8)$$

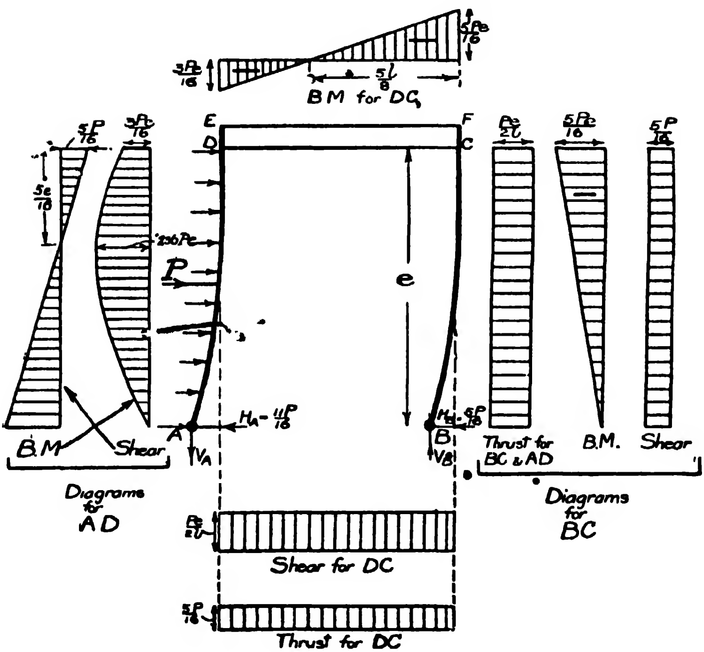


Fig. 72.

We also have as before $H_A + H_B = P$ .... (9)

$$\therefore H_A = \frac{11P}{16} \quad \text{.........................} \quad (10)$$

$$H_B = \frac{5P}{16} \quad \text{.........................} \quad (11)$$

The shear and B.M. diagrams for B C and A D then come as shown in Fig. 72. Those for B C will be followed without further explanation; the maximum B.M. in A D will occur at the point of zero shear which occurs at a distance of $\frac{11e}{16}$ from A, so that taking moments about this point we get

$$M_{max} = H_A \cdot \frac{11e}{16} - \frac{P}{e} \cdot \frac{11e}{16} \cdot \frac{11e}{32}$$

$$= \frac{121 \, P e}{512} \doteq \cdot 236 \, P e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The thrust diagrams for B C and A D will be the same except that they are reversed in sign.

The B.M., Shear, and Thrust diagrams for D C come as shown.

(3) *Columns of same Length and Stiffness; isolated Force P on Column between Top and Bottom.*—This case seldom arises in practice, but may be considered here as one of the possible conditions.

By similar reasoning to that in the previous case, we have, Fig. 73,

$$\frac{H_B \cdot e^3}{3 \, E \, I} = \frac{H_A \cdot e^3}{3 \, E \, I} - \frac{P \, x^2}{2 \, E \, I}\left(e - \frac{x}{3}\right) \quad \ldots\ldots\ldots \quad (13)$$

$$\therefore H_A - H_B = \frac{P \, x^2}{2 \, e^2}\left(3 - \frac{x}{e}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

or if $x = a \, e$

$$(H_A - H_B) = \frac{P \, a^2}{2}(3 - a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

We also have as before

$$H_A + H_B = P$$

$$\therefore H_A = \frac{P}{2}\left\{(1 + \frac{1}{2}(3 \, a^2 - a^3)\right\} \quad \ldots\ldots\ldots \quad (16)$$

$$H_B = \frac{P}{2}\left\{(1 - \frac{1}{2}(3 \, a^2 - a^3)\right\} \quad \ldots\ldots\ldots \quad (17)$$

The B.M., Shear, and Thrust diagrams then come as shown in the figure.

In the case in which P acts at the mid-point of A E, *i.e.*,

$$a = \frac{e}{2}$$

$$H_A = \frac{21\,P}{32}$$

$$H_B = \frac{11\,P}{32}$$

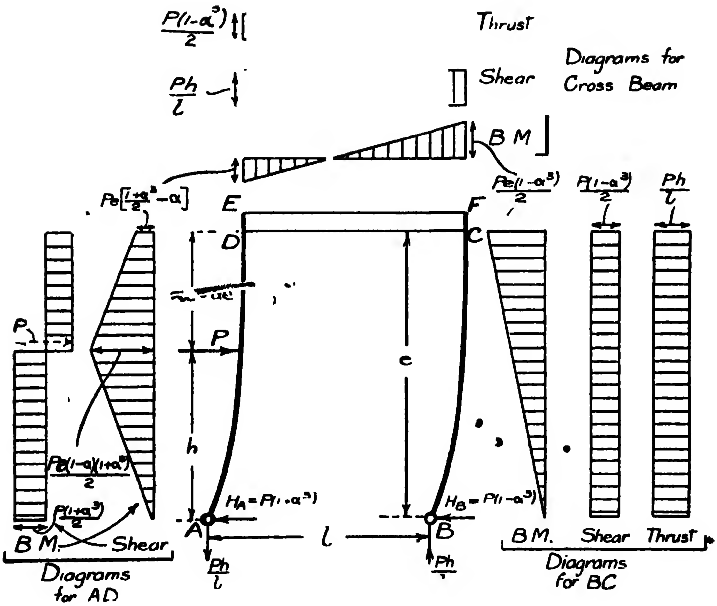


Fig. 73.

The maximum B.Ms. in the cross-beam then come equal to $\frac{7\,P\,e}{16}$ and $\frac{-\,P\,e}{12}$ respectively at the right and left side.

(4) *Columns of different Lengths and Stiffness.*—We will consider only the case of the load at the top; the same method is applicable to the other cases.

Let I_A, I_B be the moments of inertia of the two columns. Then deflection of column BC, Fig. 74, $= \dfrac{H_B e_B^3}{3 E I_B}$ and that of AD $= \dfrac{H_A e_A^3}{3 E I_A}$ so that since these deflections must be equal we get

$$\frac{H_A}{H_B} = \frac{I_A e_B^3}{I_B e_A^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

As a rough approximation, which amounts to an assumption of

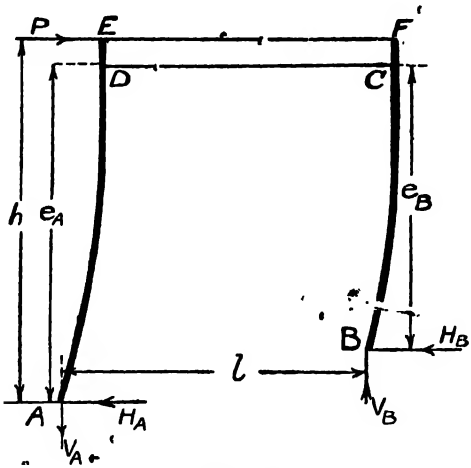


Fig. 74.

equal loads upon the columns, and the use of Euler's formula, we may take $\dfrac{I_A}{I_B} = \dfrac{e_A^2}{e_B^2}$, so that we should then get $\dfrac{H_A}{H_B} = \dfrac{e_A}{e_A}$.

(b) Columns fixed at Ends.—(1) *Columns of same Length and Stiffness; Single Force P at Top.*—In this case we get from symmetry the points G, J of contraflexure of the columns at their mid points, Fig. 75. The curves of B.M., Shear, and Thrust then come as shown in the figure, and will be followed without further explanation. It will be clear that the present case is exactly similar to the corresponding one with hinged ends, the columns being of half their previous length.

(2) *Columns of same Length and Stiffness; uniformly distributed Load on one Column.*—As in the previous case, the point of inflexion of the column B C must from symmetry be half-way up the columns (Fig. 76).

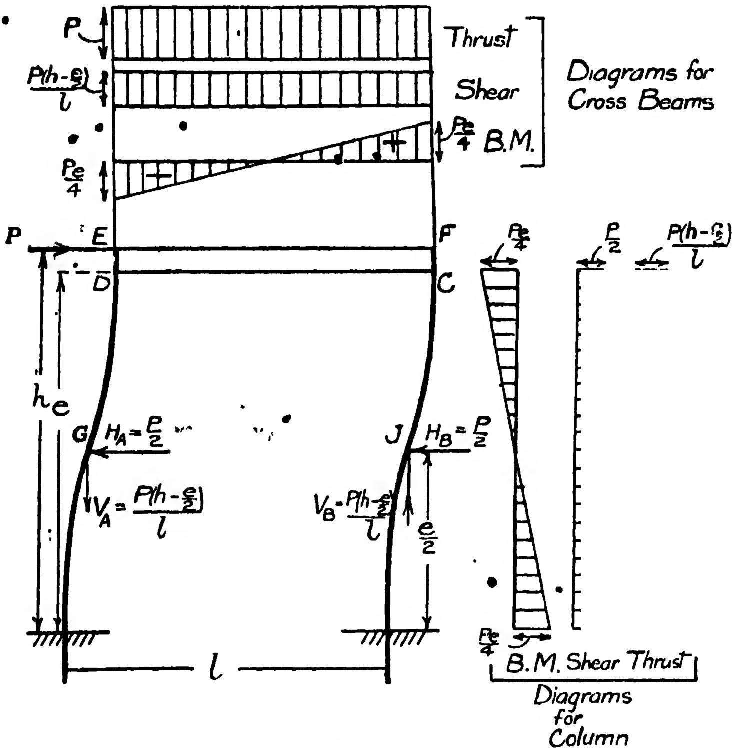


Fig. 75.

The deflection of J C can be found by considering the B.M. diagram and applying Mohr's Theorem. The area of each Δ is $\frac{H_B e^2}{8}$, so that, taking moments about the point C, we get:

$$E I \,.\, \delta_a = \frac{H_B e^2}{8} \cdot \frac{2e}{3} = \frac{H_B e^3}{12} \quad \ldots\ldots\ldots \quad (19)$$

From a consideration of the dotted line in the B.M. diagram it will be seen that the two Δ_s may be regarded as the difference between the usual Δ for a cantilever, with B the free end and a rectangle of height M_B, the bending moment at B, the area of which will act at half depth.

$$\therefore \text{ We may write } E\,I\;\delta_C = \frac{H_B\,e^3}{3} - \frac{M_B\,e^2}{2} \quad \ldots\ldots\ldots\ldots\ldots(20)$$

Coming to the column A D, the point of inflexion will not be at half depth, since the loading is not symmetrical.

Using the rule we have just investigated of treating the column as a cantilever with a constant moment in addition to that due to the load, we have

$$E\,I\,.\,\delta_D = \frac{H_A\,e^3}{3} - \frac{P\,e^3}{8} - \frac{M_A\,e^2}{2} \quad \ldots\ldots\ldots\ldots(21)$$

And putting in (19) $H_B = P - H_A$ and equating to (21) we get

$$\frac{(P - H_A)}{12} = \frac{H_A}{3} - \frac{P}{8} - \frac{M_A}{2\,e} \quad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

We have next to determine the value of M_A. This is obtained by remembering that the slope at the end D is zero since the cross-beam is rigid, and that we have, therefore, the same condition to satisfy as in the case of fixed or built-in beams, viz., that the total area of the B.M. diagram must be zero. Now this B.M. diagram will be made up of a Δ of height $H_A\,e$ at D and area $\frac{H_A\,.\,e^2}{2}$; of a parabola of height $-\frac{P\,e}{2}$ at D and area $-\frac{1}{3}\,.\,\frac{P\,e^2}{2}$ and a rectangle of height $- M_A$ and area $- M_A\,.\,e$.

$$\therefore \text{ we have } \frac{H_A\,e^2}{2} - \frac{1}{3}\,\frac{P\,e^2}{2} - M_A\,e = 0 \quad \ldots\ldots\ldots\ldots(23)$$

$$\text{or } \quad M_A = \frac{H_A\,.\,e}{2} - \frac{P\,e}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Putting this into (22) we get:

$$H_A = \frac{3\,P}{4},\; H_B = \frac{P}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

Then $$M_A = \frac{3\,Pe}{8} - \frac{Pe}{6} - \frac{5\,Pe}{24}. \qquad .(26)$$

Now let us determine the distance y of the point of inflexion from the base.

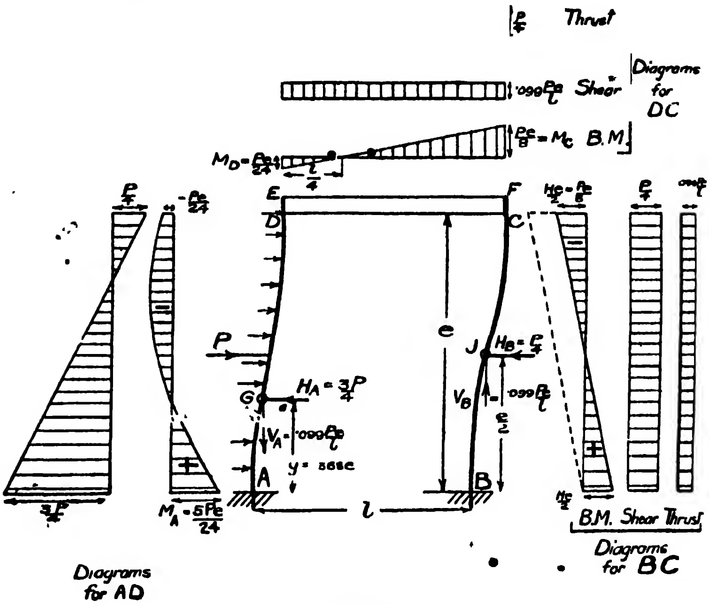


Fig. 76.

At a distance y from A the B.M. is given by

$$M_y = M_A - H_A \cdot y + \frac{P}{e} \cdot \frac{y^2}{2}.$$

$$\therefore \quad \frac{5\,Pe}{24} - \frac{3\,Py}{4} + \frac{Py^2}{2e} = 0 \qquad \ldots\ldots\ldots\ldots(27)$$

The solution of this quadratic gives $\frac{y}{e} = \cdot 368$

The B.M. at the end D is given by :

$$M_D = M_A - H_A \cdot e + \frac{Pe}{2}$$

$$= \frac{5 P e}{24} - \frac{3 P e}{4} + \frac{P e}{2}$$

$$= - \frac{P e}{24} \quad \text{...(28)}$$

The B.M. diagrams then came as shown in Fig. 76. To get the values of V_B or V_A we take moments about, say, G.

$$\text{Then} \left(P - \frac{P}{4}\right) \times \left(\frac{e}{2} - \cdot 368\, e\right) = V_B \, . \, l$$

$$i.e. \quad V_B = \frac{\cdot 099\, P e}{l} \quad \text{.................................(29)}$$

*Cross-beam not Rigid.

If the cross-beam is not rigid, the slope of the cross-beam at the ends has to be considered, and the slope of the columns at the connections to the cross-beam must be the same as the slope at the ends of the cross-beam if the connections are rigid. We will restrict our considerations to columns of equal length and stiffness.

(*a*) **Columns Pin-jointed at their Ends.**—(1) Single Load at Top.—The B.M., Shear, and Thrust diagrams in this case will be the same as in the corresponding case with the rigid cross-beam, Fig. 71. The deflections of the columns will, however, be increased by amounts $\tan \theta_C \, . \, e$ and $\tan \theta_D \, . \, e$, which will in this case from symmetry be equal.

If I_B is the moment of inertia of the cross-beam, we can get a value of the end slopes by a consideration of Mohr's imaginary cable such as is employed in obtaining the proof of the Theorem of Three Moments, we shall then get

$$E\, I_B\, l \tan \theta_C = \frac{-M_A\, l^2}{6} - \frac{2\, M_B\, l^2}{6} \text{*}$$

* *See A, p.* 252. We there prove:

$$E\, I\, l_1 \tan \theta = S_1 y_1 - \frac{M_A\, l_1^2}{6} - \frac{2\, M_B\, l_1^2}{6}$$

In the present case $S_1 = 0$ because there is no transverse load on the beam, and to get this into our present form put $I = I_B$, $l_1 = l$, $M_A = M_C$ and $M_B = M_D$. A similar equation with reverse signs will result if we take moments about the other support and applies equally well to the present case of a beam with ends partly fixed as in the case of part of a continuous beam.

$$i.e. \quad \tan \theta_C = \frac{-l}{6\,E\,I_B}(M_D + 2\,M_C)$$

$$\text{similarly } \tan \theta_D = \frac{l}{6\,E\,I_B}\{M_C + 2\,M_D\} \qquad .(30)$$

In our present case $\theta_C = \theta_D$ and $M_C = \frac{P\,e}{2}$, $M_D = \frac{-P\,e}{2}$

$$\tan \theta_C = \tan \theta_D = \frac{P\,e\,l}{12\,E\,I_B} \quad \ldots\ldots\ldots\ldots\ldots(31)$$

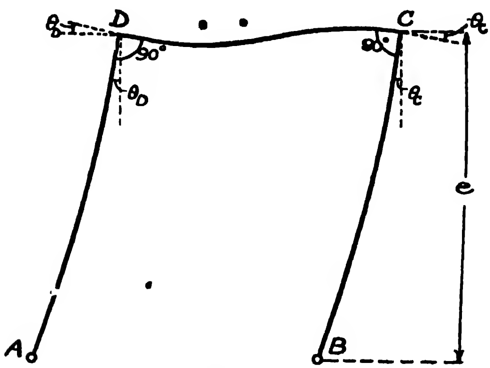


Fig. 77.

We reverse the signs which would be given by the above equations, because in this case we reckon θ_C and θ_D as positive, whereas in the above treatment they would be taken as negative. If, therefore, I is the moment of inertia of each column we get:

$$\delta_C = \delta_D = \frac{P\,e^3}{6\,E\,I} + \frac{P\,e^2\,l}{12\,E\,I_B}$$

$$= \frac{P\,e^2}{6\,E}\left\{\frac{e}{I} + \frac{l}{2\,I_B}\right\} \quad \ldots\ldots\ldots\ldots\ldots(32)$$

Professor Morley* has suggested a simplification of this and subsequent similar formulæ by writing $\frac{I_B}{l} \div \frac{I}{e}$ = some coefficient

* Morley's *Theory of Structures* (Longmans).

say s; s gives a kind of relative stiffness of the cross-beam and column and we will call it the *stiffness coefficient.*

Then $$\delta_C = \delta_D = \frac{P e^3}{6 E I}\left\{1 + \frac{1}{2 s}\right\} \quad \ldots\ldots\ldots\ldots(33)$$

(2) UNIFORMLY DISTRIBUTED LOAD.—In this case the slopes will not be the same at each end because the loading on each column is not symmetrical.

Referring to Fig. 72, we have $M_C = H_B \cdot e$ and $M_D = \frac{P e}{2} - H_B \cdot e = \frac{P e}{2} - (P - H_B) e = e\left(H_B - \frac{P}{2}\right)$

$\therefore$ From equation (30), reversing the signs for the reason previously explained,

$$\tan \theta_C = \frac{l e}{6 E I_B}\left(H_B - \frac{P}{2} + 2 H_B\right)$$

$$= \frac{l e}{12 E I_B}(6 H_B - P) \quad \ldots\ldots\ldots\ldots(34)$$

$$\tan \theta_D = \frac{l e}{6 E I_B}(H_B + 2 H_B - P)$$

$$= \frac{- l e}{6 E I_B}(3 H_B - P) \quad \ldots\ldots\ldots\ldots(35)$$

$$\therefore \quad \delta_C = \frac{H_B e^3}{3 E I} + e \tan \theta_C$$

$$= \frac{H_B e^3}{3 E I} + \frac{l e^2}{12 E I_B}(6 H_B - P) \quad \ldots\ldots(36)$$

$$\therefore \quad \delta_D = \frac{(P - H_B) e^3}{3 E I} - \frac{P e^3}{8 E I} - \frac{l e^2}{6 E I_B}(3 H \;\; - P) \ldots(37)$$

Equating these two deflections we get

$$H\left\{\frac{2 e^3}{3 E I} + \frac{l e^2}{E I_B}\right\} = P\left\{\frac{e^3}{3 E I} - \frac{e^3}{8 E I} + \frac{l e^2}{4 E I_B}\right\}$$

Dividing through by $\frac{e^2}{1}$ and simplifying, this gives

$$H_B = P \times \frac{\frac{5 e}{24 I} + \frac{l}{4 I_B}}{\frac{2 e}{3 I} + \frac{l}{I_B}}$$

$$= \frac{P}{8} \times \frac{\frac{5e}{I} + \frac{6l}{I_B}}{\frac{2e}{I} + \frac{3l}{I_B}} \quad \text{........ ....(37a)}$$

We may simplify this as before, thus getting

$$H_B = \frac{P}{8} \times \frac{(5s + 6)}{(2s + 3)} \quad \text{....................(38)}$$

When the stiffness coefficient s is very large this gives $H_B = \frac{5P}{16}$, the value that we obtained on page 175, and when s is negligibly small, *i.e.*, the cross-beam has practically no rigidity compared with that of the columns, we get $H_B = \frac{P}{4}$.

(3) Isolated Force P on Column between Top and Bottom.—Referring to Fig. 73, we shall have in this case

$$M_C = H_B \cdot e$$

$$M_D = P \cdot x - H_A \cdot e$$

$$= e\,(a\,P - P + H_B)$$

$$= e\,\{H_B - P\,(1 - a)\} \quad \text{..............(39)}$$

$$\therefore \tan\theta_C = \frac{l\,e}{6\,E\,I_B}\left\{H_B - P\,(1 - a) + 2\,H_B\right\}$$

$$= \frac{l\,e}{6\,E\,I_B}\left\{3\,H_B - P\,(1 - a)\right\} \quad \text{............. ......(40)}$$

$$\tan\theta_D = \frac{-\,l\,e}{6\,E\,I_B}\left\{H_B + 2\,H_B - 2\,P\,(1 - a)\right\}$$

$$= \frac{-\,l\,e}{6\,E\,I_B}\left\{3\,H_B - 2\,P\,(1 - a)\right\} \quad \text{.................(41)}$$

$$\therefore \quad \delta_C = \frac{H_B\,e^3}{3\,E\,I} + \frac{e^2\,l}{6\,E\,I_B}\left\{3\,H_B - P\,(1 - a)\right\} \quad \text{........(42)}$$

$$\delta_D = \frac{(P - H_B)\,e^3}{3\,E\,I} - \frac{P\,a^3\,e^3}{3\,E\,I} - \frac{e^2\,l}{6\,E\,I_B}\left\{3\,H_B - 2\,P\,(1 - a)\right\} \text{(43)}$$

Equating these we get:

$$H_B \left\{ \frac{2\,e^3}{3\,E\,I} + \frac{e^2\,l}{E\,I_B} \right\} = P \left\{ \frac{(1 - a^3)\,e^3}{3\,E\,I} + \frac{P\,(1 - a)\,e^2\,l}{2\,E\,I_B} \right\}$$

$$H_B = \frac{P \left\{ \frac{(1 - a^3)\,s}{3} + \frac{(1 - a)}{2} \right\}}{\frac{2\,s}{3} + 1}$$

$$= \frac{P}{2} \times \frac{\{2\,(1 - a^3)\,s + 3\,(1 - a)\}}{2\,s + 3} \quad (44)$$

If $a = \frac{1}{2}$, this gives

$$H_B = \frac{P}{2} \times \left(\frac{\frac{7\,s}{4} \times \frac{3}{2}}{2\,s + 3} \right)$$

$$= \frac{P}{8} \times \frac{(7\,s + 6)}{(2\,s + 3)} \quad \text{.....................} (45)$$

If s is very great this gives as before $H_B = \frac{7\,P}{16}$, and, if s is negligibly small, it gives $H_B = \frac{P}{4}$.

(b) Columns fixed at Ends.—(1) Single Force P at Top.—In this case, as in the corresponding case with rigid cross-beams, the value of H_B and H_A must each, from symmetry, be equal to $\frac{W}{2}$, but the lack of rigidity of the cross-beam will alter the position of the point of contraflexure. In this case

$$M_C = H_B\,(e - y) = \frac{W}{2}\,(e - y)\,;\; M_D = \frac{-\,W}{2}\,(e - y)$$

And from equation (30)

$$\tan\theta_C = \frac{l}{6\,E\,I_B} \left\{ \frac{W}{2}\,(e - y) \right\} \quad \text{..............} (46)$$

Again, considering the slope of the beam B C, we shall have

$$E\,I\,\tan\theta_J = H_B \left(e\,y - \frac{e^2}{2} \right) \text{.......................} (47)$$

This is obtained as follows: At distance z from B,

$$M = H_B\,(y - z)$$

$$\therefore \quad \mathrm{E\,I} \tan\theta = \int \mathrm{M}\, dz = \mathrm{H_B}\left(yz - \frac{z^2}{2}\right) + \mathrm{C}$$

Slope $=$ o at B where $z =$ o $\quad \therefore \mathrm{C} =$ o

$$\therefore \quad \mathrm{E\,I} \tan\theta_c = \mathrm{H_B}\left[\left(yz - \frac{z^2}{2}\right)\right]_0^e$$

$$= \mathrm{H_B}\left(ey - \frac{e^2}{2}\right)$$

$\therefore$ equating (46) and (47), and putting $\mathrm{H_B} = \dfrac{\mathrm{W}}{2}$ we get

$$\frac{l}{6\,\mathrm{E\,I_B}}\left\{\frac{\mathrm{W}}{2}(e-y)\right\} = \frac{\mathrm{W}}{2\,\mathrm{E\,I}}\left(ey - \frac{e^2}{2}\right)$$

$$\frac{l}{6\,\mathrm{I_B}}(e-y) = \frac{2ey - e^2}{2\,\mathrm{I}}$$

$$y\left(\frac{e}{\mathrm{I}} + \frac{l}{6\,\mathrm{I_B}}\right) = \frac{e}{2}\left(\frac{e}{\mathrm{I}} + \frac{l}{3\,\mathrm{I_B}}\right)$$

$$y = \frac{\dfrac{e}{2}\left(\dfrac{e}{\mathrm{I}} + \dfrac{l}{3\,\mathrm{I_B}}\right)}{\dfrac{e}{\mathrm{I}} + \dfrac{l}{6\,\mathrm{I_B}}}$$

$$= \frac{\dfrac{2e}{2}(3s+1)}{(6s+1)}$$

$$= e \times \frac{(3s+1)}{(6s+1)} \quad \ldots\ldots\ldots\ldots(48)$$

If s is very great, this gives as before $y = \dfrac{e}{2}$, and if s is negligibly small, we get as we should expect $y =$ o, *i.e.*, there will be no point of contraflexure; this corresponds to a hinged connection at B and C.

2. Uniformly distributed Load.—In this case we cannot assume as in the previous case that $\mathrm{H_B}$ and $\mathrm{H_A}$ are each equal to $\dfrac{\mathrm{W}}{2}$.

Referring to Figs. 76 and 78, we have from equation (30)

$$\tan \theta_C = \frac{l}{6\,E\,I_B}\,(M_D + 2\,M_C)$$

$$\tan \theta_D = \frac{-\,l}{6\,E\,I_B}\,(M_C + 2\,M_D)$$

Now $\quad M_C = H_B \,.\, e - M_B$ ....(49)

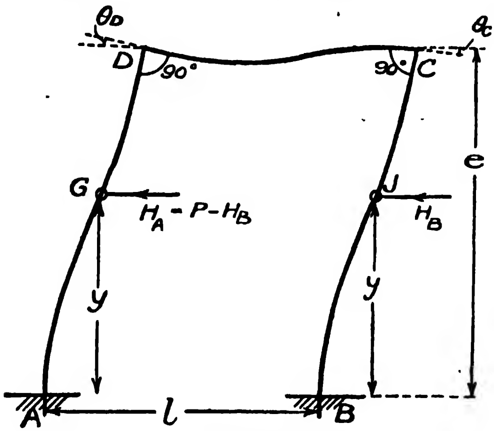


Fig. 78.

And $\quad - M_D = \frac{P\,e}{2} - H_B \,.\, e - M_A$.(50)

$$\therefore\quad E\,I_B \tan \theta_C = \frac{l}{6}\left\{3\,H_B \,.\, e + M_A - 2\,M_B - \frac{P\,e}{2}\right\} \text{...(51)}$$

$$E\,I_B \tan \theta_D = \frac{-\,l}{6}\left\{3\,H_B \,.\, e + 2\,M_A - M_B - P\,e\right\} \text{...(52)}$$

Now consider the slope of the columns. We get as before for the column B C

$$E\,I \tan \theta_C = M_B \,.\, e - \frac{H_B\,e^2}{2} \qquad .(53)$$

Then for the columns A D, noting that the slope at A is zero, we get

$$E\,I \tan \theta_D = M_A \,.\, e - \frac{(P - H_B)\,e^2}{2} + \frac{P\,e^2}{6}$$

$$M_A\, e + \frac{H_B\,e^2}{2} - P\,e^2 \qquad .(54)$$

∴ Equating the values of tan θ_C and tan θ_D from these equations we get

$$\frac{l}{6\,E\,I_B}\left(3\,H_B\,.\,e + M_A - 2\,M_C - \frac{P\,e}{2}\right) = \frac{e}{E\,I}\left(M_B - \frac{H_B\,e}{2}\right)$$

$$\therefore\quad 6\,s\left(M_B - H_B\,\frac{e}{2}\right) = 3\,H_B\,.\,e + M_A - 2\,M_B - P\frac{e}{2} \quad \ldots(55)$$

Similarly

$$6\,s\left(M_A + \frac{H_B\,e}{2} - \frac{P\,e}{3}\right) = -\left(3\,H_B\,.\,e + 2\,M_A - M_B - P\,e\right) \ldots(56)$$

Finally consider the deflections at C and D which we will assume equal. Integrating again from the slopes, we get

$$E\,I\,\delta_C = \frac{M_B\,e^2}{2} - \frac{H_B\,e^3}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(57)$$

$$E\,I\,\delta_D = \frac{M_A\,.\,e^2}{2} - \frac{(P - H_B)\,e^3}{6} + \frac{P}{e}\,.\,\frac{e^4}{24}$$

$$= \frac{M_A\,e^2}{2} - \frac{Pe^3}{8} + \frac{H_B\,e^2}{6} \quad \ldots\ldots\ldots\ldots\ldots(58)$$

Equating these deflections and dividing by e^2 we get

$$\frac{M_A - M_B}{2} = \frac{P\,e}{8} - \frac{H_B\,e}{3}$$

$$\text{or}\quad M_A - M_B = \frac{P\,e}{4} - \frac{2\,H_B\,e}{3} \quad \ldots\ldots\ldots\ldots(59)$$

Rearranging equations (55) and (56) we get

$$M_B\,(6\,s + 2) - M_A = \frac{-\,P\,e}{2} + 3\,H_B\,e\,(1 + s)$$

$$-\,M_B + M_A\,(6\,s + 2) = P\,e\,(1 + 2\,s) - 3\,H_B\,e\,(1 + s)$$

Subtracting, we get

$$(M_A - M_B)\,(6\,s + 3) = P\,e\left(\frac{3}{2} + 2\,s\right) - 6\,H_B\,.\,e\,(1 + s) \ldots(60)$$

Combining this with (59) we get :

$$(6\,s + 3)\left(\frac{P\,e}{4} - \frac{2\,H_B\,e}{3}\right) = P\,e\left(\frac{3}{2} + 2\,s\right) - 6\,H_B\,e\,(1 + s)$$

$$H_B\,(6 + 6\,s - 4\,s - 2) = P\left(\frac{3}{2} + 2\,s - \frac{3}{4} - \frac{3\,s}{2}\right)$$

$$H_B(2s+4) = P\left(\frac{s}{2}+\frac{3}{4}\right)$$

$$i.e. \quad H_B = \frac{P}{8} \cdot \frac{(2s+3)}{(s+2)} \quad \ldots\ldots\ldots\ldots(61)$$

If s is great, we get from this as before $H_B = \frac{P}{4}$, and if s is negligibly small, $H_B = \frac{3P}{16}$, this again corresponding to the hinged connection at B and C.

Putting this value of H_B in the above equations we get from equation (59)

$$M_A - M_B = \frac{Pe}{4} - \frac{Pe(2s+3)}{12(s+2)}$$

$$= \frac{Pe}{12}\left\{\frac{3s+6-2s-3}{s+2}\right\}$$

$$= \frac{Pe}{12}\frac{(s+3)}{(s+2)}$$

$$\therefore \quad M_A = M_B + \frac{Pe}{12}\frac{(s+3)}{(s+2)}$$

Put this result in (55), then

$$M_B(6s+2) - M_B - \frac{Pe}{12}\left(\frac{s+3}{s+2}\right) = \frac{-Pe}{2} + \frac{3Pe}{8}\frac{(2s+3)(s+1)}{s+2}$$

$$M_B(6s+1) = \frac{Pe}{24}\left\{\frac{2s+6-12(s+2)+9(2s+3)(s+1)}{s+2}\right\}$$

$$= \frac{Pe(18s^2+35s+9)}{24(s+2)}$$

$$i.e. \quad M_B = \frac{Pe}{24}\frac{(18s^2+35s+9)}{(6s+1)(s+2)} \quad \ldots\ldots\ldots\ldots(62)$$

$$\therefore \quad M = \frac{Pe}{24}\frac{(30s^2+73s+15)}{(6s+1)(s+2)} \quad \ldots\ldots\ldots\ldots(63)$$

From equation (49)

$$\therefore \quad M_0 = \frac{Pe}{8}\frac{(2s+3)}{(s+2)} - \frac{Pe}{24}\frac{(18s^2+35s+9)}{(6s+1)(s+2)}$$

$$= \frac{Pe}{24} \frac{[3(2s+3)(6s+1) - (18s^2+35s+9)]}{(6s+1)(s+2)}$$

$$= \frac{Pe}{24} \frac{s(18s+25)}{(6s+1)(s+2)} \quad \text{......................(64)}$$

From equation (50)

$$M_D = \frac{Pe}{2} - \frac{Pe}{8} \frac{(2s+3)}{(s+2)} - \frac{Pe}{24} \frac{(30s^2+73s+15)}{(6s+1)(s+2)}$$

$$= \frac{Pe}{24} \frac{[12(6s+1)(s+2) - 3(2s+3)(6s+1) - (30s^2+73s+15)]}{(6s+1)(s+2)}$$

$$= \frac{Pe}{24} \frac{s(6s+23)}{(6s+1)(s+2)} \quad \text{..................(65)}$$

Usual Rules for Design.—We have seen in the foregoing treatment that several of the cases lead to a complicated analysis, and in dealing with cases of multiple portals consisting of portals arranged one above the other, or one after the other, the analysis becomes more complicated still. When it is remembered that in practice we shall seldom get the perfect rigidity of the joints that we have assumed, it will be seen that the following simpler treatment for purposes of design will probably be the most suitable:—

(*a*) *Columns hinged at their Bases.*—Take each column as resisting equally the total horizontal force, *i.e.*, divide the total horizontal force by the number of columns, and consider this force as acting at the base of the columns.

(*b*) *Columns fixed at their Bases.*—Take the point of inflexion of each column at the mid point, *i.e.*, divide up the total horizontal force as in case (*a*) above, and consider it as acting halfway up the column.

FRAMED PORTALS.

In the case of framed portals, the values of the horizontal reactions of the columns may be found as in the previous treatment, and are usually taken in practice as suggested above. The stresses in the bracing can then be found by means of the reciprocal figure construction; additional bracing, indicated in

dotted lines in Figs. 79 and 80, being assumed to enable the diagrams to be drawn. This additional bracing does not alter the

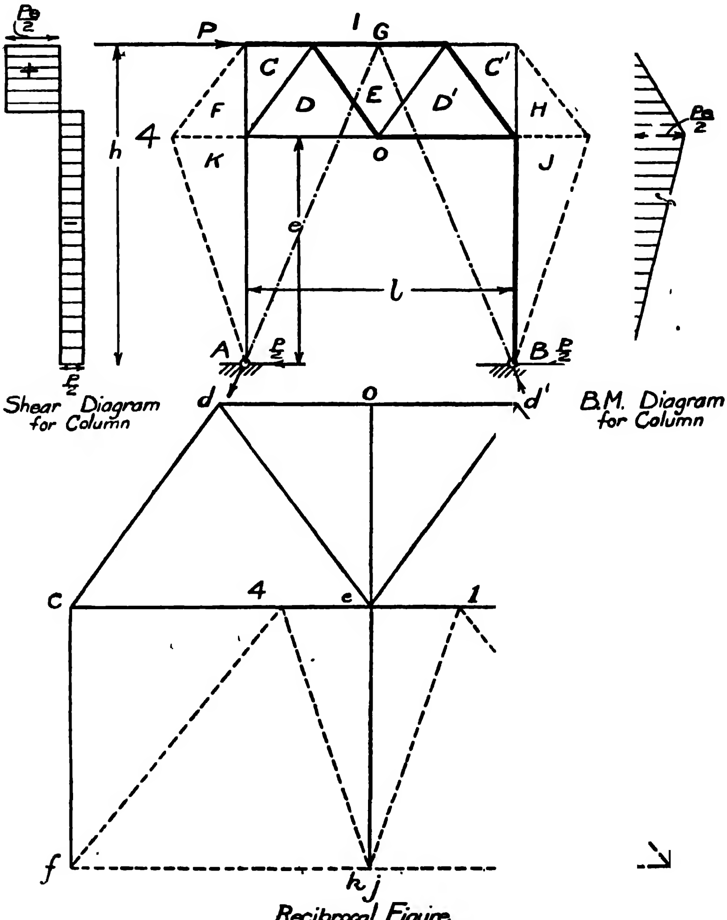


Fig. 79.—*Framed Portals.*

stresses in the remaining bars of the frame, but serves to replace the beam action of the columns by an equivalent truss action.

Ends of Columns Pin-jointed (Fig. 79).—In this case we assume each horizontal component of the reactions to be equal to $\frac{P}{2}$; the two reactions will then intersect at the mid point of the

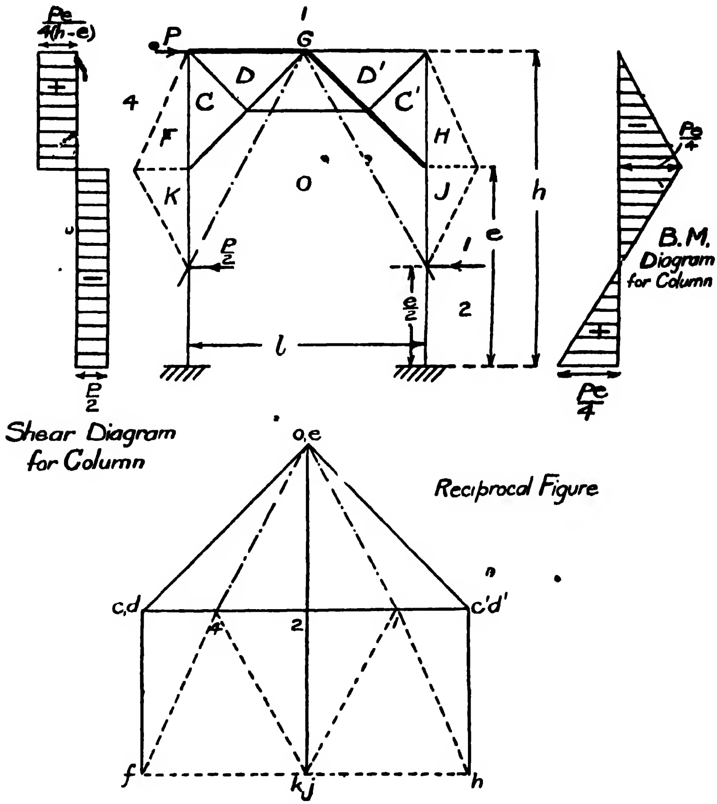


Fig. 80.—*Framed Portals.*

top of the portal. Then setting out 4, 1 to represent the force P, and drawing parallels to A G and G'B, we get the point O upon the reciprocal figure, and the reciprocal figure can then be drawn without difficulty, the bars shown in thick lines being in

compression. Instead of drawing parallels to the reactions we might set up $e\,o = \frac{P\,h}{2}$. The B.M. and shear diagrams then come as shown in the figure.

Ends of Columns Fixed (Fig. 80)—In this case the procedure is practically the same as in the previous case, but the points of inflexion, through which the reactions are taken as acting, are taken as mid way between the column bases and the cross bracing. as shown. The reciprocal figure is then drawn without difficulty, and it will be noted that the stresses in the bars C D, C′ D′, and the horizontal cross brace are zero.

***Point of Inflexion of Fixed Columns with Framed Portals.**—In the above treatment we have taken the point of inflexion as half-way up the unsupported portion of the column. Although most designers consider this sufficiently accurate for practical purposes, the point of inflexion can be investigated more accurately as follows.

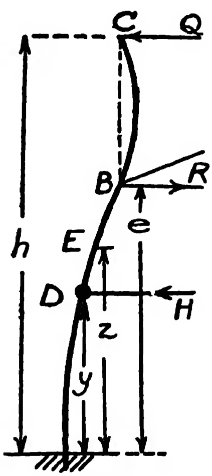


Fig. 81.

We assume that the cross bracing is sufficiently rigid to maintain the cross frame horizontal, *i.e.*, the points B, C, Fig. 81, will remain on the same vertical line, so that they will have the same deflection.

There will be a resultant force Q at C (*i.e.*, the difference between the force P and the stress in the bracing) and a force R at B.

Treating these as the forces on a cantilever, we have for any point E between B and the fixed end, where δ is the deflection.

$$M_E = E\,I\,\frac{d^2\,\delta}{d\,z^2} = Q\,(h - z) - R\,(e - z) \qquad .(1)$$

Integrating once we get,

$$E\,I\,\frac{d\,\delta}{d\,z} = Q\left(h\,z - \frac{z^2}{2}\right) \quad R\left(e\,z \;-\frac{z^2}{2}\right) \;\ldots\ldots\quad .(2)$$

Integrating again we get for B

$$\therefore \quad E\,I\,\delta_B = \left[Q\left(\frac{h\,z^2}{2} - \frac{z^3}{6}\right) - R\left(\frac{e\,z^2}{2} - \frac{z^3}{6}\right)\right]_0^e$$

$$= \frac{e^2}{2}\left\{Q\left(h - \frac{e}{3}\right) - R\left(e - \frac{e}{3}\right)\right\} \quad \text{.........(3)}$$

Between B and C

$$M = E\,I\,\frac{d^2\,\delta}{d\,z^2} = Q\,(h - z)$$

$$E\,I\,\frac{d\,\delta}{d\,z} = \int M\,d\,z = Q\left(h\,z - \frac{z^2}{2}\right) + C \quad \text{............(4)}$$

when $z = e$, this must give the same result as equation (2).

$$\therefore \quad Q\left(h\,e - \frac{e^2}{2}\right) - R\left(e^2 - \frac{e^2}{2}\right) = Q\left(h\,e - \frac{e^2}{2}\right) + C$$

$$\therefore \quad C = -\frac{R\,e^2}{2}$$

$\therefore$ integrating (4) again we get

$$E\,I\,.\,\delta = Q\left(\frac{h\,z^2}{2} - \frac{z^3}{6}\right) - \frac{R\,e^2\,z}{2} + C_1 \text{............(5)}$$

when $z = e$, this must agree with (3)

$$\therefore \frac{e^2}{2}\left\{Q\left(h - \frac{e}{3}\right) - 2\,R\left(\frac{e}{3}\right)\right\} = \frac{e^2}{2}\left\{Q\left(h - \frac{e}{3}\right) - \frac{R\,e}{2}\right\} + C_1$$

$$\therefore \quad C_1 = \frac{R\,e^3}{6}$$

Putting this into our equation for the deflection at the point C we shall get

$$E\,I\,\delta_C = \frac{Q\,h^2}{2}\left(h - \frac{h}{3}\right) - \frac{R\,e^2\,h}{2} + \frac{R\,e^3}{6} \text{............(6)}$$

Since the deflections at B and C are equal we have

$$\frac{e^2}{2}\left\{Q\left(h - \frac{e}{3}\right) - R\left(\frac{2\,e}{3}\right)\right\} = \frac{Q\,h^2}{2}\left(h - \frac{h}{3}\right) - \frac{R\,e^2}{2}\left(h - \frac{e}{3}\right)$$

$$Q\left\{\frac{h^3}{3} - \frac{h\,e^2}{2} + \frac{e^3}{6}\right\} = R\left(\frac{e^2\,h}{2} - \frac{e^3}{2}\right)$$

$$\therefore \quad \frac{Q}{R} = \frac{3\,e^2\,(h - e)}{(2\,h^3 - 3\,h\,e^2 + e^3)} = \frac{3\,e^2\,(h - e)}{(h - e)\,(2\,h^2 + 2\,h\,e - e^2)}$$

$$= \frac{3\,e^2}{(2\,h^2 + 2\,h\,e - e^2)} \quad \ldots\ldots(7)$$

Now by moments about the point of contraflexure, we have $Q\,(h - y) = R\,(e - y)$

$$\therefore \quad \frac{Q}{R} = \frac{(e - y)}{(h - y)}$$

$$\therefore \quad \frac{e - y}{h - y} = \frac{3\,e^2}{2\,h^2 + 2\,h\,e - e^2}$$

$$\therefore \; y\,(2\,h^2 + 2\,h\,e - e^2 - 3\,e^2) = e\,(2\,h^2 - 2\,e\,h - e^2) - 3\,e^2\,h$$

$$y\,(2\,h^2 + 2\,h\,e - 4\,e^2) = e\,(2\,h^2 - e\,h - e^2)$$

$$y = \frac{e}{2}\left(\frac{2\,h^2 - e\,h - e^2}{h^2 + h\,e - 2\,e^2}\right)$$

$$= \frac{e}{2}\,\frac{(2\,h + e)\,(h - e)}{(h + 2\,e)\,(h - e)}$$

$$= \frac{e}{2}\,\frac{(2\,h + e)}{(h + 2\,e)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

For $e = \frac{h}{2}, y = \frac{5}{8}\,e$

$e = \frac{2\,h}{3}, y = \frac{4\,e}{7}$

$e = h$ (the extreme case), $y = \frac{e}{2}$

We see therefore that for the relations between e and h that are likely to arise in practice, y lies between $\frac{5\,e}{8}$ and $\frac{e}{2}$, so that the value $y = \frac{e}{2}$ is never seriously in error.

Knee-braced Portals.—In the knee-braced portal the cross-beam E F (Fig. 82) is considered as pin-jointed at E and F, and braces D K C L are provided to give lateral rigidity.

ENDS PIN-JOINTED.—If the columns are of equal length and

rigidity, each horizontal reaction will be equal to $\frac{P}{2}$ for a load P at E; for other cases this is approximately correct and is usually assumed in design although a nearer approximation will be obtained by taking the values of H_A and H_B as previously obtained for a rigid cross-beam with rigid connection.

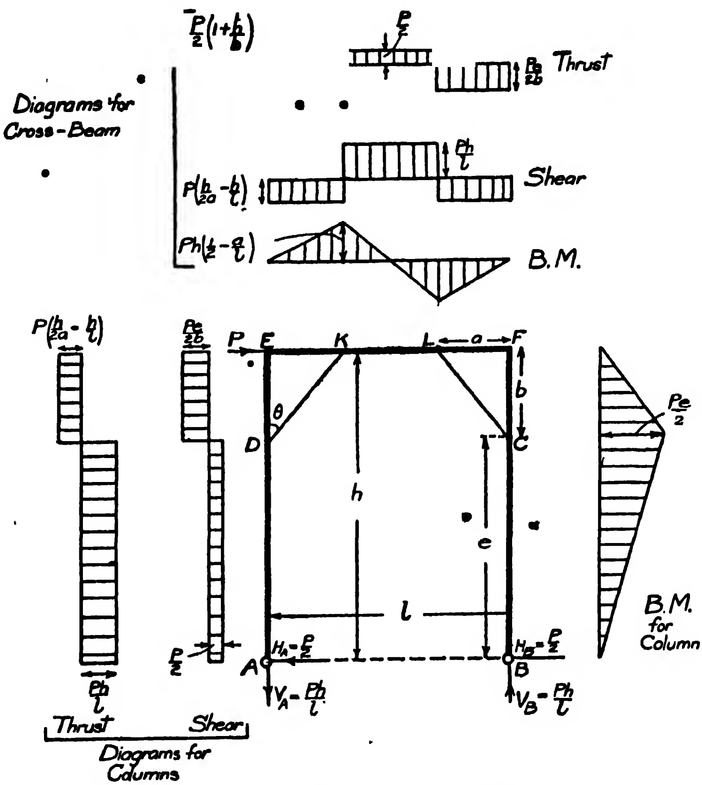


Fig. 82.—*Knee-braced Portals.*

The B.M., &c., diagrams for the columns up to D and C come as shown in Fig. 82 and will be followed without further explanation.

We will next consider the stresses in the braces. The

horizontal component of the tension in K D must be equal, by moments about E to $\frac{H_A}{2}$

$$\therefore \underset{KD}{f} = H_A \cdot \frac{h}{b} \operatorname{cosec} \theta \text{ (tension)}$$

$$H_A \cdot \frac{h}{b} \cdot \frac{\sqrt{a^2 + b^2}}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$= \frac{P}{2} \cdot \frac{h\sqrt{a^2 + b^2}}{a\,b} \text{ on our assumption}$$

$$\text{Similarly } \underset{CL}{f} = H_B \cdot \frac{h\sqrt{a^2 + b^2}}{a\,b} \text{ (compression)}$$

$$= -\underset{KD}{f} \text{ on our assumption}$$

Next consider the shear, thrust, and B.M. upon the cross-beam, which is strictly a continuous beam of three spans.

Now the force in F C will be

$$f = V_B - \underset{CL}{f} \cdot \cos \theta$$

$$= \frac{P\,h}{l} - \frac{P}{2} \cdot \frac{h\sqrt{a^2 + b^2}}{a\,b} \times \frac{b}{\sqrt{a^2 + b^2}}$$

$$= P\left(\frac{h}{l} - \frac{h}{2\,a}\right)$$

$$= -P\left(\frac{h}{2\,a} - \frac{h}{l}\right), \text{ compression}$$

$$\text{or } P\left(\frac{h}{2\,a} - \frac{h}{l}\right) \text{ tension} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This will be the downward reaction at F on the cross-beam so that the B.M. at L will be equal to

$$M_L = R_F \times a = -P\,h\left(\frac{1}{2} - \frac{a}{l}\right) \quad \ldots\ldots\ldots\ldots(3)$$

$$\text{Similarly } M_K = +P\,h\left(\frac{1}{2} - \frac{a}{l}\right)$$

The B.M. will vary uniformly between L and K, so that we get the complete B.M. diagram as shown.

Next consider the shear on the cross-beam.

From F to L the shear will be equal to R_F,

$$i.e. = -P\left(\frac{h}{2a} - \frac{h}{l}\right)$$

From L to K the shear will be equal to

$$-R_F + f_{LC}\cos\theta = V_B = \frac{Ph}{l}$$

From K to E the shear will be equal to

$$-R_F + f_{LC}\cos\theta - f_{DK}\cos\theta = -R_F$$

The complete shear diagram comes therefore as shown.

Finally consider the thrust in E F.

$$\text{Thrust in FL} = f_{FL} = -\frac{H_B \cdot e}{b} \text{ (by moments about C)}$$

$$= -\frac{Pe}{2b}$$

$$\text{Thrust in KL} = -\frac{Pe}{2b} + f_{LC}\sin\theta$$

$$= f_{KL} = -\frac{Pe}{2b} + \frac{Ph}{2b}$$

$$\frac{P}{2}\left(\frac{h-e}{b}\right)$$

$$\text{Thrust in KE} = f + f\sin\theta$$

$$\frac{P}{2} + \frac{Ph}{2b}$$

$$= \frac{P}{2}\left(1 + \frac{h}{b}\right)$$

The complete thrust diagram then comes as shown.

Flexible Braces.—If the braces are made of tie-bar section and are unable to carry compression stresses, the leeward or

right-hand brace L C, Fig. 83, will go out of action, and the structure becomes structurally determinate, and the whole horizontal reaction comes at the point A, *i.e.*, $H_A = P$, $H_B = 0$.

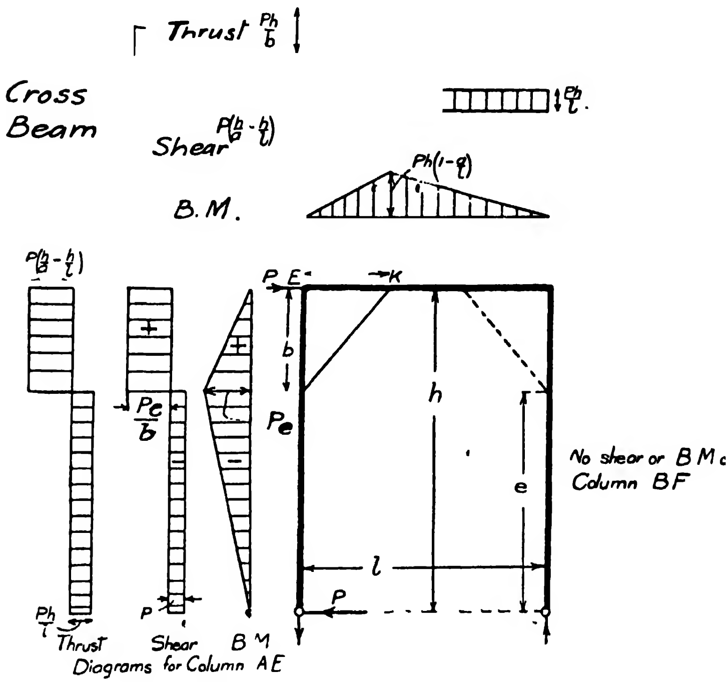


Fig 83.—Knee-braced Portals.

In this case the stress in the tie will have twice its previous value, *i e.*, $f_{DK} = \frac{Ph\sqrt{a^2 + b^2}}{ab}$ and the B.M., shear, and thrust diagrams come as shown in Fig. 83.

COLUMNS FIXED AT ENDS.—In this case the usual assumption is to take the points of inflexion at the mid points of A D and B C, and the stresses will be the same as in the previous case as if we measure the distance *h* and *e* to the points of inflexion

instead of to the bases. Greater accuracy for other cases may be found by using the results previously obtained for the different cases.

*** Portals with Vertical Loading on Cross-beam.**—When cross-beams carry vertical loads and the connections with the columns are rigid, the columns become subjected to bending moments and deflect somewhat as indicated in Fig. 84, (*a*) representing hinged bases and (*b*) indicating fixed bases.

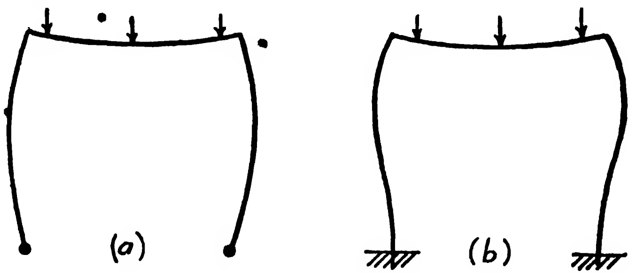


Fig 84.

The general procedure to find the thrust in this case is similar to that for rigid arches, and consists in equating to zero the deflection of one support with reference to another.

COLUMNS HINGED AT BASES.—We showed on p. 122 that the horizontal deflection of the point B, Fig. 85, with reference to the point A, is equal to $\int_A^B \frac{M\,y\,d\,s}{E\,I}$, and that if we call M_1 the bending moment which would occur if the end B were free to move we get $M = M_1 - H \cdot y$.

$$\therefore \text{ we get} \int_A^B \frac{(M_1 - H\,y)\,y\,d\,s}{E\,I} = 0$$

$$\text{or } H = \frac{\int_A^B \frac{M_1\,y\,d\,s}{E\,I}}{\int_A^B \frac{y^2\,d\,s}{E\,I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

This is the same formula as for the arch, and assuming that E is constant we may write

$$H = \frac{\int_A^B \frac{M_1 y\,ds}{I}}{\int_A^B \frac{y^2\,ds}{I}} \quad \text{.....................} (1a)$$

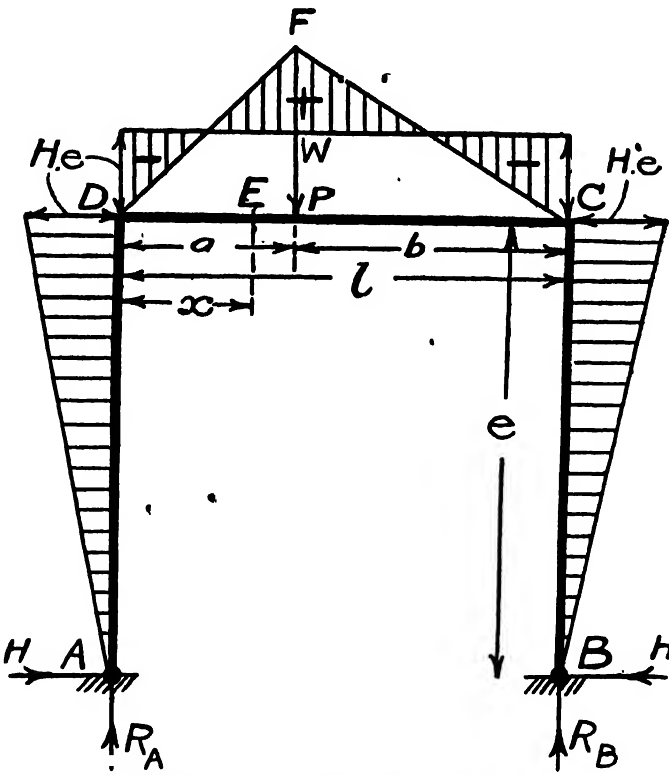


Fig. 85.—Portals with Vertical Loading.

The integrations in the present case have to be performed in three steps; taking as before I_B as the moment of inertia of the

cross-beam and I as that of the columns (assumed as of the same section) we get

$$\int_A^B \frac{y^2\,ds}{\mathrm{I}} = \int_A^D \frac{e^2\,de}{\mathrm{I}} + \int_D^C \frac{e^2\,dx}{\mathrm{I_B}} + \int_C^B \frac{e^2\,de}{\mathrm{I}}$$

(remembering that y represents vertical distances)

$$i.e.\ \int_A^B \frac{y^2\,ds}{\mathrm{I}} = \frac{e^3}{3\,\mathrm{I}} + \frac{e^2\,l}{\mathrm{I_B}} + \frac{e^2}{3\,\mathrm{I}}$$

$$= e^2\left(\frac{2\,e}{3\,\mathrm{I}} + \frac{l}{\mathrm{I_B}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$= \frac{}{\mathrm{I}}\left(\frac{2}{3} + \frac{l\,\mathrm{I}}{e\,\mathrm{I_B}}\right)$$

$$= \frac{e^3}{\mathrm{I}}\left(\frac{2}{3} + \frac{1}{s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Now for the columns, $\mathrm{M_1} = 0$

$$\therefore \int_A^B \frac{\mathrm{M_1}\,y\,ds}{\mathrm{I}} = \int_D^C \frac{\mathrm{M_1}\,e\,dx}{\mathrm{I_B}} = \frac{e}{\mathrm{I_B}}\int_D^C \mathrm{M_1}\,dx$$

but (assuming $\mathrm{I_B}$ constant), $\int_D^C \mathrm{M_1}\,dx = \mathrm{S_1}$, the area of the free B.M. diagram for the cross-beam

$$\therefore \int_A^B \frac{\mathrm{M_1}\,y\,ds}{\mathrm{I}} = \frac{e\,\mathrm{S_1}}{\mathrm{I_B}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\mathrm{H} = \frac{\dfrac{e\,\mathrm{S_1}}{\mathrm{I_B}}}{\dfrac{e^3}{\mathrm{I}}\left(\dfrac{2}{3} + \dfrac{1}{s}\right)} = \frac{\mathrm{S_1}}{\dfrac{e\,l\,e\,\mathrm{I_B}}{\mathrm{I}\,.\,l}\left(\dfrac{2}{3} + \dfrac{1}{s}\right)}$$

$$= \frac{\mathrm{S_1}}{e\,l\,s\left(\frac{2}{3} + \frac{1}{s}\right)}$$

$$= \frac{3\,\mathrm{S_1}}{e\,l\,(2\,s + 3)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The B.M. at the top of each column will then be equal to H e and the B.M. diagram will be as shown in the figure.

Isolated Load W.—In this case the free B.M. diagram is as shown in Fig. 85 and $S_1 = \frac{1}{2} \cdot l \cdot \frac{W \cdot a b}{l}$

$$\therefore \quad H = W \times \frac{3 a b}{2 e l (2 s + 3)} \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

$$\therefore \quad \text{Max. B.M. on column} = H e = \frac{3 W a b}{2 l (2 s + 3)}$$

Uniform Load.—In this case the free B.M. diagram is a parabola and $S_1 = \frac{2}{3} \cdot \frac{l \cdot W l}{8} = \frac{W l^2}{12}$

$$\therefore \quad H = W \times \frac{l}{4 e (2 s + 3)} \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

$$\therefore \quad \text{Max. B.M. on column} = \frac{W l}{4 (2 s + 3)}$$

COLUMNS FIXED AT ENDS.—In this case the analysis becomes much more complicated if we attempt to deal with it as an arch with fixed ends; we therefore proceed as follows.

Let M_D and M_C be the 'end bending moments' on the cross-beams at D and C; these will be equal to the bending moments at the ends of the columns. We can then equate the deflections at the points D and C.

We then get $\quad E I \delta_D = -\frac{H e^3}{3} + \frac{M_D e^2}{2}$

$$E I \delta_C = \frac{H e^3}{3} - \frac{M_C e^2}{2}$$

If these are equal, $M_C + M_D = \frac{4 H e}{3} \qquad \ldots\ldots\ldots\ldots\ldots(1)$

Next consider the slopes of the columns at the tops.

We have then

$$E I \tan \theta_D = \int_o M \, d z$$

$$\int^e \left\{ M_D - H (e - z) \right\} d z + C$$

$$= M_D \,.\, e - H e^2 + \frac{He^2}{2} + 0, \text{ because slope}$$

$= 0$ when $z = 0$

$$= M_D e - \frac{H e^2}{2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Similarly

$$E\,I \tan \theta_C = - M_C e + \frac{H e^2}{2} \quad \dots\dots\dots\dots\dots\dots\dots (3)$$

$$\therefore \quad E\,I\,(\tan \theta_D - \tan \theta_C) = (M_D + M_C)\,e - H e^2 \dots\dots\dots (4)$$

i.e. $E\,(\tan \theta_D - \tan \theta_C)$

$$= \frac{H e^2}{3\,I} \text{ (from (1))} \quad \dots\dots\dots\dots\dots (5)$$

Again consider the slopes at the ends of the cross beam; we have as in the footnote on p. 182.

$$E\,I_B\,l \tan \theta_C = S_1 y_1 - \frac{M_D\,l^2}{6} - \frac{2\,M_C\,l^2}{6} \dots\dots (5a)$$

Similarly $\quad E\,I_B\,l \tan \theta_D = S_1\,(l - y_1) - \dfrac{M_C\,l^2}{6} - \dfrac{2\,M_D\,l^2}{6} \quad (6)$

Adding we get

$$E\,I_B\,l\,(\tan \theta_C - \tan \theta_D) = S_1\,l - \frac{M_C\,l^2}{2} - \frac{M_D\,l^2}{2} \quad \dots\dots\dots (7)$$

$$\textit{i.e.} \quad E\,(\tan \theta_C - \tan \theta_D) = \frac{S_1}{I_B} - \frac{(M_D + M_C)\,l}{2\,I_B} \dots\dots\dots\dots\dots (8)$$

$$= \frac{S_1}{I_B} - \frac{2\,H\,e\,l}{3\,I_B} \text{ [from (1)]} \quad \dots\dots (9)$$

We now equate (5) and (9) and reverse signs; since the slopes of the beam and column are at 90° to each other we then get

$$\frac{H e^2}{3\,I} = \frac{S_1}{I_B} - \frac{2\,.\,H\,e\,l}{3\,I_B}$$

$$\textit{i.e.} \quad \frac{H\,e\,l}{3\,I_B}\left(\frac{I_B\,.\,e}{l\,.\,I} + 2\right) = \frac{S_1}{I_B}$$

$$\textit{i.e.} \quad H = \frac{3\,S_1}{e\,l\,(s + 2)} \quad \dots\dots\dots\dots\dots\dots\dots (10)$$

The values of M_C and M_D can then be found from equations (1), (2), and (6), but it is rather better to substitute actual values

for a given case than to obtain the resulting rather complicated formulæ.

We then have

$$M_A = M_D - He \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$M_B = M_C - He \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Symmetrical Loading.—In this case we get considerable simplification in the formulæ because $M_C = M_D$.

Then (1) gives $M_D = M_C = \frac{2\,H\,e}{3}$, so that $M_A = M_B = \frac{-\,H\,e}{3}$

∴ The point of contraflexure of the columns is at distance $\frac{l}{3}$ from the bases.

Uniform Loading.—In this case $S_1 = \frac{2\,l}{3} \cdot \frac{W\,l}{8} = \frac{W\,l^2}{12}$

$$\therefore \quad H = \frac{W\,l}{4\,e\,(s + 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$\therefore \quad M_D = M_C = \frac{W\,l}{6\,(s + 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$M_A = M_B = -\frac{W\,l}{12\,(s + 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Isolated Load.—In this case $S_1 = \frac{l}{2} \cdot \frac{W\,a\,b}{l}$

$$\therefore \quad H = \frac{3\,W\,a\,b}{2\,e\,l\,(s + 2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Also $y_1 = \frac{l + a}{3}$, taking the load as nearer D than C*.

Now from (2) $$E \tan \theta_D = \frac{M_D\,e}{I} - \frac{H\,e^2}{2\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

" (5) $$E \tan \theta_D = \frac{W\,a\,b}{?} \cdot \frac{(l+a)}{3 \cdot l\,I_B} - \frac{l}{6\,I_B}(M_D + 2\,M_C) \ldots (18)$$

$$\therefore \left(M_D - \frac{H\,e}{2}\right)\frac{e}{I} = \frac{l}{6\,I_B}\left\{\frac{W\,a\,b\,(l + a)}{l^2} - (M_D + 2\,M_C)\right\} \ldots (19)$$

$$\therefore 6\,s\left(M_D - \frac{H\,e}{2}\right) = \frac{W\,a\,b\,(l + a)}{l^2} - M_D + 2\,M_C \quad \ldots\ldots\ldots (20)$$

* See A, p. 207, and reverse b and a.

In any numerical case we calculate H first, and then put the various values of H, e, a, b, l, s into (1) and (20), thus getting two equations, from which M_C and M_D can be obtained without much difficulty.

CURVES FOR UNIFORM LOADING. — For uniform loading

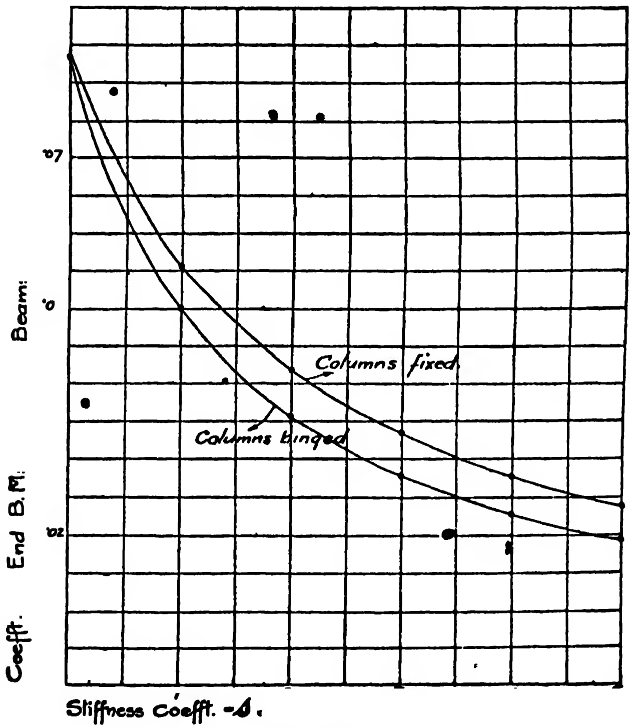


Fig. 86.

Fig. 86 gives the coefficients by which W l has to be multiplied to give the maximum bending moments upon the columns and the end bending moments for the cross-beam for various values of the *stiffness coefficient* s. It will be noted that for $s = 0$, which, of course, corresponds to columns infinitely stiff compared with

the beam, we get for both methods of fixing a coefficient $\cdot 083 = \frac{1}{12}$. *This is the* familiar result for a beam fixed horizontally at the ends.

Applications to Reinforced Concrete Construction.—In ordinary steel frame construction for buildings it is usual to neglect the rigidity of the joints, so that the columns are not considered as subjected to bending moments, and the cross-beams are not considered as fixed at the ends. This is to some extent justified on the ground that in steelwork there is always some play in the joints.

In reinforced concrete construction, however, the portals are monolithic, and the bending moments upon the columns should be considered as well as the reverse bending moments upon the cross-beams if cracks are to be avoided. We shall deal further with this under continuous portals, which we next consider.

CONTINUOUS PORTALS.

Sheds and like buildings often consist of a series of portals arranged one after the other somewhat as indicated in Fig. 87, and are called continuous portals.

HORIZONTAL LOADING.—In this case it is usual to take the horizontal resistance of each column as equal. This would be true if all the columns are identically equal, and the load were applied at the top only, as shown in the figure. Therefore the horizontal force at the base of each column $= \frac{P}{n}$ where n is the number of columns. For other cases we could proceed in the same general manner as in previous cases, but the analysis usually becomes too complicated for practical application. For fixed ends the points of contraflexure are taken at the mid depth, and so this case corresponds to that shown, the length of column being taken as $\frac{e}{2}$ instead of e.

Let G be the centroid of the group of columns; then it is reasonable to suppose that the force on each column is proportional to the distance from the centroid.

Let u be the reaction at unit distance from the centroid: then $V_1 = u\,x_1$; $V_2 = u\,x_2$; $V_3 = u\,x_3$; $V_4 = u\,x_4$ and so on, the reaction for the n^{th} column being $u\,x_n$,

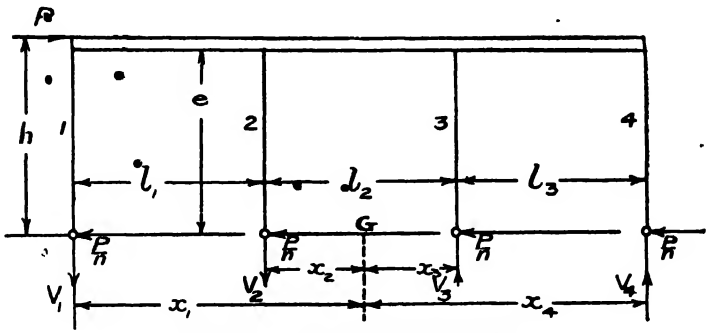


Fig. 87.—Continuous Portals.

Now the moment about G of the reactions must be equal to the moment of P about G.

$$\textit{i.e.}\ V_1 x_1 + V_2 x_2 + \ldots\ V_n x_n = P h$$

$$\textit{i.e.}\ u\,(x_1^2 + x_2^2 + \ldots\ x_n^2) = P h$$

$$\textit{i.e.}\ u = \frac{P h}{\Sigma\, x^2} \qquad .(1)$$

From this the value of the various reactions can be found.

NUMERICAL EXAMPLE.—*A continuous portal of* 5 *spans each* 18 *feet has a load of* 2500 *lbs. applied at the top of the cross-beam, which is* 2 *feet deep, and the bottom of which is* 20 *feet from the ground. Taking each column as fixed at the ends, find the vertical force in each column and the maximum B.M.*

In this case e = 20 ft., and once the ends are fixed the effective length will be 10 ft.

Now in this case n = 6.

$$\therefore\ H = \frac{2500}{6} = 417 \text{ lbs. nearly.}$$

$$\therefore\ \text{Max. B.M. on each column} = 417 \times 10 = 4170 \text{ ft. lbs.}$$

$$x_1 = x_6 = 45\ ;\ x_2 = x_5 = 27\ ;\ x_3 = x_4 = 9$$

$$\therefore \quad \Sigma x^2 = 2 \times 9^2 (1^2 + 3^2 + 5^2)$$
$$= 2 \times 81 \times 35$$
$$\therefore \quad u = \frac{2500 \times 22}{2 \times 81 \times 35} = 8{\cdot}82 \text{ lbs.}$$

$$\therefore \quad R_6 = -R_1 = 45 \times 8{\cdot}82 = 397 \text{ lbs.}$$
$$R_5 = -R_2 = 27 \times 8{\cdot}82 = 238 \text{ lbs.}$$
$$R_4 = -R_3 = 9 \times 8{\cdot}82 = 79{\cdot}3 \text{ lbs.}$$

It is interesting to note the similarity between the above formula (1) and the ordinary bending formula $\frac{f}{y} = \frac{M}{I}$. In the latter case $\frac{f}{y}$ is the stress at unit instance from the Neutral Axis and corresponds to u; Σx^2 represents the moment of inertia of the columns per square inch of section and Ph is the bending moment.

Vertical Loading; Application to Reinforced Concrete Construction.—The most important application of continuous portals with vertical loading occurs in reinforced concrete construction, in which the calculation of the bending moments upon the columns caused by the monolithic nature of the structures is of great importance, because the stresses resulting from these bending moments are often considerably in excess of those due to the direct load.

The analysis of the bending moments becomes very troublesome except for the cases where the spans are equal and the loading is uniformly distributed, but this is the case which occurs most frequently in practice.

This subject is very carefully dealt with in *Reinforced Concrete Design*, by Faber and Bowie (Arnold), and for a full treatment the reader should consult this book; we will just give their approximate treatment of the case of two spans, one of which carries only the dead load W_2, Fig. 88, and the other of which carries a combined dead and live load W_1, this being one of the worst cases that can arise.

Let θ_1, θ_2, and θ_3 be the inclinations at the tops of the

columns, which will also be the slopes at the three supports of the cross-beam.

The approximation consists in neglecting the bending resistance of the columns compared with the cross-beams, *i.e*, it consists in assuming that the slopes of the beam will be the same

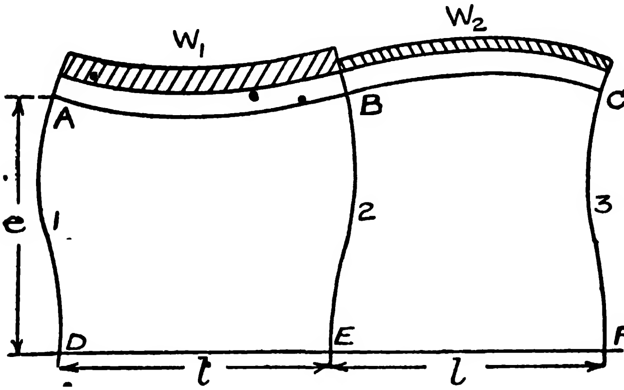


Fig. 88.—*Continuous Portals with Vertical Load.*

as if it just rested upon supports and that the columns will not alter such slopes.

Then we shall have

$$E\,I_B\ \ l\tan\theta_1 = S_1 y_1 - \frac{2\,M_1\,l^2}{6} - \frac{M_2\,l^2}{6}$$

In this case $S_1 = \frac{2\,l}{3} \cdot \frac{W_1\,l}{8} = \frac{W_1\,l^2}{12}$ and $y_1 = \frac{l}{2}$, also $M_1 = 0$

and $M_2 = \frac{(W_1 + W_2)\,l}{16}$ (from the Theorem of Three Moments)

$$\therefore\ E\,I_B\,l\tan\ \theta_1 = \frac{W_1\,l^3}{24} - \frac{(W_1 + W_2)\,l^3}{96}$$

$$\therefore\ E\tan\theta_1 = \frac{(3\,W_1 - W_2)\,l^2}{96\,I_B} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\text{Similarly } E\, I_B\, l \tan\theta_2 = -\left(S_1 y_1 - \frac{2\, M_2\, l^2}{6} - \frac{M_1\, l^2}{6}\right)$$

$$= -\frac{W_1\, l^3}{24} + \frac{(W_1 + W_2)\, l^3}{48}$$

$$= -\frac{(W_1 - W_2)\, l^3}{48}$$

$$\therefore E \tan\theta_2 = -\frac{(W_1 - W_2)\, l^2}{48\, I_B} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

If the ends of the columns are hinged, it follows as in the recent examples that the B.M. diagrams for the columns will be the triangles with apices at the base. Treating the B.M. diagram as the load, the slope will be the corresponding shear; and since the centroid of the Δ acts at $\frac{2\,e}{3}$ from the base, the imaginary reactions, which correspond to the end shears, will be twice as great at the top as at the bottom.

$$\text{Total imaginary load} = \frac{M \times e}{2}$$

$$\therefore E\, I \tan\theta = \frac{2}{3} \cdot \frac{M\, e}{2} = \frac{M\, e}{3}$$

$$\textit{i.e.}\ M = \frac{3\, I}{e} \cdot E \tan\theta$$

$$M_1 = \frac{3\,(3\, W_1 - W_2)\, l^2 \,.\, I}{96\, I_B \,.\, e}$$

$$= \frac{(3\, W_1 - W_2)\, l}{\dfrac{32\, I_B \,.\, e}{l \,.\, I}}$$

$$= \frac{(3\, W_1 - W_2)\, l}{32\, s} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\text{Similarly } M_2 = \frac{-(W_1 - W_2)\, l}{16\, s} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

M_3 will be less than M_1, and so need not be calculated.

WIND STRESSES IN TALL BUILDINGS.

In order to give lateral rigidity in skeleton frames for tall buildings, some form of lateral bracing is adopted; the resulting frame is then a complicated kind of portal, the accurate analysis of which would be practically impossible.

We will give the usual method of calculation adopted in the case of sway bracing, Fig. 89; the same calculations, as far as

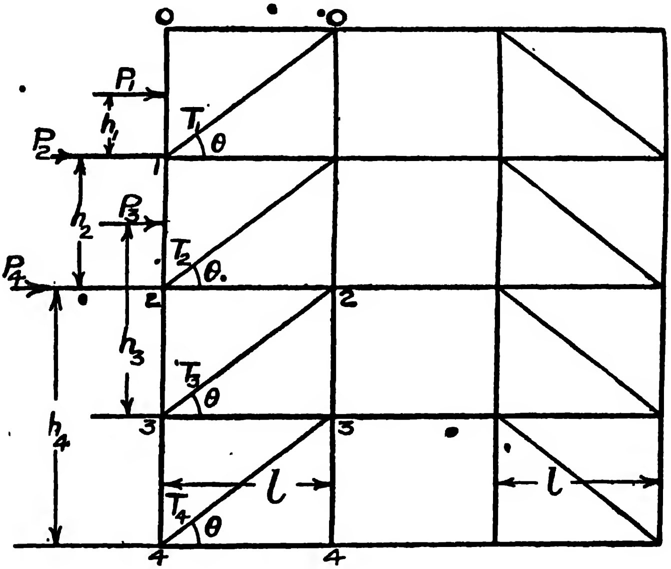


Fig. 89.—*Sway Bracing.*

the effect upon the column load goes, apply to other forms of bracing.

In the illustration it is assumed that wind bracing is only provided between the exterior columns and the next ones.

The effect of the wind is to decrease the load on the exterior or windward columns and to increase that on the interior or leeward ones.

At the line 1, 1, the decrease and increase in load in exterior and interior columns respectively is equal to

$$V_1 = \frac{P_1 h_1}{l}$$

At the line 2, 2 it is equal to

$$V_2 = \frac{P_2 h_2}{l}$$

and so on, the value at the base being

$$V_4 = \frac{P_4 h_4}{l}$$

The stresses in the diagonals will be given respectively by :

$$T = S \sec \theta$$

where S is the shearing force in the bay under consideration. If all the bays are equal. *i.e.*, $P_2 = 2 P_1$, $P_3 = 3 P_1$, and $P_4 = 4 P_1$, we get

$$T_1 = \frac{P_1 \sec \theta}{2}$$

$$T_2 = \frac{3 P_1 \sec \theta}{2}$$

$$T_3 = \frac{5 P_1 \sec \theta}{2}$$

$$T_4 = \frac{7 P_1 \sec \theta}{2}$$

If instead of sway bracing we have knee bracing, or some other form of portal bracing, we can get the stresses in the manner dealt with for the simple portals, the resultant forces being taken as P_1, P_2, P_3, P_4 respectively, and each column giving equal horizontal resistance.

Knee bracing is, as a rule, not economical, because it causes heavy bending moments in the columns and cross-beams.

CHAPTER X.

SECONDARY STRESSES IN STRUCTURES.

WITHIN recent years considerable attention has been given to the consideration of 'secondary stresses;' these may be defined as the additional stresses brought into play because the structures as actually erected do not agree with the theoretical forms upon which are based the formulæ for the ordinary calculations. In framed girders, for instance, we always assume that the centre lines of the members intersect at the nodes and that the joints are hinged; in riveted joints also we assume that the centre line of the rivets always passes down the centroid line of the section. Any departure from these theoretical conditions—and such departures nearly always occur in practice—induce secondary stresses, which are in some cases just as severe as the ordinary or primary stresses. Our aim in design should be to eliminate these secondary stresses as much as possible; when they cannot be eliminated wholly, we ought to allow for them in our calculations. Some authorities have urged that the actual stresses which occur will not be so great as the sum of the calculated primary and secondary stresses, and that therefore we might take some such rule as: effective stress = primary stress + $\frac{1}{2}$ secondary stress. This contention is probably true in many cases since a slight 'give' of the parts will reduce the secondary stresses; the only satisfactory method of settling the problem is an exhaustive series of tests carried out in a scientific manner such as those which Mr. Batho* has been making at McGill University, Toronto, upon the stresses in angle and built-up sections.

Secondary Stresses due to Eccentric Rivet Connections.—The commonest case of this kind is that of a ⊥ or L section used as a tie or strut in a framed structure. For most

* *Proceedings of the Canadian Society of Civil Engineers*, 1912.

sizes in common use, the centre line of the section—*i.e.*, that passing through the centroid—is too near to the flange for the centre of the rivet holes to be made to agree with it, the result being that such sections are almost invariably loaded.

NUMERICAL EXAMPLE.—*Take, for a first example, a* 4″ × 4″ × ½″ T-*bar, Fig.* 90, *used, say, as the rafter of a roof truss.*

The centre line of the rivets connecting this bar to the other parts of the structure is usually at a distance of 2¼ in. from the flange.

If the length of rafter between supports be taken as 6 ft. and we treat the ends as pin-jointed, then, according to the usual formulæ —such, for instance, as those tabulated in Dorman, Long & Co.'s *Pocket Companion*—the safe load on this bar is 11·1 tons. Most designers would consider this section as quite satisfactory if the calculated load were not greater than this.

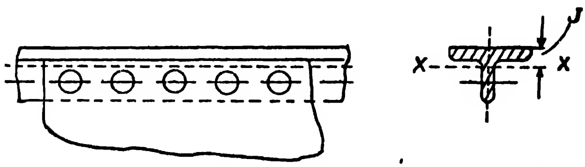


Fig. 90.—*Secondary Stresses in Riveted Joints.*

Now, let us look at the properties of this section as given in the standard tables. They are :

$$\text{Area A} = 3{\cdot}76 \text{ sq. in.}$$
$$\text{Distance J to centroid} = 1{\cdot}16 \text{ in.}$$
$$I_{xx} = 5{\cdot}40$$

In the use of this bar, therefore, the load is transmitted at a distance $= 2{\cdot}25 - 1{\cdot}16 = 1{\cdot}09$ in. out of centre.

The bending moment due to this is $M = 11{\cdot}1 \times 1{\cdot}09 = 12{\cdot}1$ inch tons.

To a first approximation the compressive bending stress caused in this manner $= \dfrac{My}{I} = \dfrac{12{\cdot}1 \times (4 - 1{\cdot}16)}{5{\cdot}40} = \dfrac{12{\cdot}1 \times 2{\cdot}84}{5{\cdot}40}$

$$= 6{\cdot}4 \text{ tons per sq. in. approximately}$$

$$\therefore \text{ Total stress} = \frac{W}{A} + \frac{My}{I} = \frac{11{\cdot}1}{3{\cdot}76} + 6{\cdot}4 = 2{\cdot}9 + 6{\cdot}4$$
$$= 9.3 \text{ tons per sq. in}$$

This is considerably above the 'safe' figure.

It will also be noted that the secondary stress of 6·4 tons per sq. in. is more than twice the primary stress of 2·9 tons per sq. in.

A more accurate formula for calculating the secondary stress, which allows for the additional eccentricity due to the deflection of the bar, is

$$\text{Stress}^* = \frac{My}{1 - \frac{Wl^2}{10\,E}} = \frac{12{\cdot}1 \times 2{\cdot}84}{5{\cdot}40 - \frac{11{\cdot}1 \times 72 \times 72}{10 \times 13{,}000}} = 6{\cdot}9 \text{ tons per sq. in.}$$

E in this formula is, as usual, Young's Modulus.

Eccentric Rivet Connections in Framed Structures.—As we have previously indicated, secondary stresses often arise due to the fact that the centre lines of the members of a framed structure do not intersect at the nodes. In the case shown in Fig. 90*a*, for example, a bending moment equal to $R \times y$ or $P \times x$ is caused by this fact. This bending moment will be distributed between the members making up the joint if the joint is rigid, and will be carried by the flexible members equally if it is not. A consideration of the slopes of the members such as we have already made for the cross-beams of portals shows that the bending moment for a rigid joint will be divided among the members proportionately to the value of $\frac{I}{l}$ for the members, where I is the moment of inertia and l their length. The calculation

* This formula is known as Johnson's formula, and is obtained by adding to the original eccentricity the additional amount due to the deflection. It can be obtained as an approximate solution of the strut with a lateral load. [See, for instance, Morley's *Strength of Materials*.] A rather more accurate, but still approximate, formula for a strut with an eccentric load would be one with 8 instead of 10 as the constant; this can be deduced as follows:—Deflection of a beam with a uniform B.M. $= \delta = \frac{M\,l^2}{8\,E\,I}$; and $f = \frac{My}{I}$; then if x is the eccentricity

$$f = \frac{W(x+\delta)y}{I} = \frac{Wxy}{I} + \frac{Wfl^2}{8\,E\,I}$$

$$\therefore f\left(1 - \frac{Wl^2}{8\,E}\right) = \frac{Wxy}{I}$$

$$\therefore f = \frac{My}{1 - \frac{Wl^2}{8\,E\,I}}$$

M being the bending moment neglecting the deflection. The riveted joint at the ends of the strut will tend to diminish this when there is, as usual, more than one rivet.

then follows quite simply, as illustrated in the following numerical example :—

NUMERICAL EXAMPLE.—*Take the example in which* $P = 35{,}600$ *and* $x = 7{\cdot}5$ *in., the diagonals being at* 90 *degrees to each other. For the chord members take two angles,* $6'' \times 4'' \times \frac{3}{8}''$ *and for the diagonals take two angles* $4'' \times 3'' \times \frac{5}{16}''$, *the lengths of the chord members being* 12 *ft. and of the diagonal members* 8 *ft.*

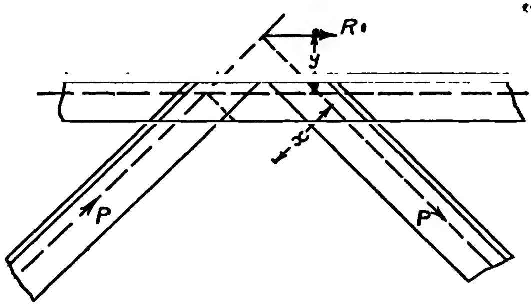


Fig. 90*a.*—*Secondary Stresses.*

We get the following values from the tables of standard sections :—

$6'' \times 4'' \times \frac{3}{8}''$ angle. $\quad I = 13{\cdot}2. \quad A = 3{\cdot}61. \quad y = 1{\cdot}91.$

$4'' \times 3'' \times \frac{5}{16}''$ angle. $\quad I = 3{\cdot}31. \quad A = 2{\cdot}09. \quad y = 1{\cdot}24.$

In this case $M = 7{\cdot}5 \times 35{,}600 = 267{,}000$ in lbs.

Dividing this in the proportion of $\frac{13{\cdot}2}{12}$ to $\frac{3{\cdot}31}{8}$ we get B.M. on each chord $= \frac{1}{2} \times \frac{267{,}000 \times 1{\cdot}1}{1{\cdot}51} = 97{,}200$ in lbs. ; B.M. on each diagonal $= \frac{1}{2}(267{,}000 - 972{,}000) = 84{,}900$ in lbs.

$\therefore$ Bending stress in chords $= \frac{My}{I} = \frac{97{,}200}{26{\cdot}4} \times (6 - 1{\cdot}91)$

$= 15{,}000$ lbs. per sq. in. nearly.

Bending stress in diagonals $= \frac{84{,}900}{6{\cdot}62} \times (4 - 1{\cdot}24)$

$= 35{,}400$ lbs. per sq. in. nearly.

These are in each case about the limit of the safe stresses independently of the primary stresses, and if, as is probable, the actual stresses are somewhat less than these, we require very careful experimental investigation of the problem.

Design of Riveted Joints to avoid Secondary Stresses.—Mr. E. W. Pittman, in a paper read before the Engineering Society of Western Pennsylvania (*Proceedings*, 1909–10), gives some interesting sketches of riveted joints as commonly designed and as they should be designed to minimise secondary stresses and to get equal loads on the rivets.

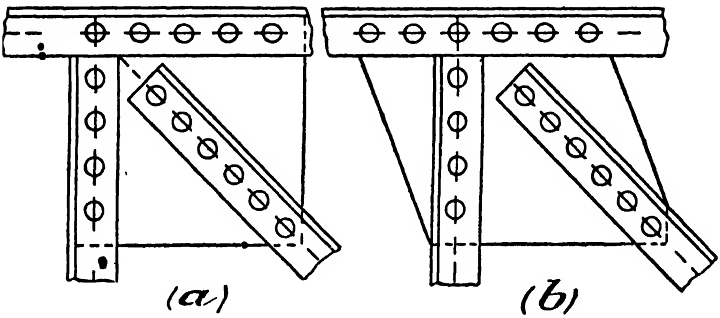


Fig. 91.—*Secondary Stresses in Riveted Joints.*

Fig. 91 shows a joint in an N girder. (*a*) is a common arrangement; (*b*) is the more correct variation.

Fig. 92 (*a*) shows a common arrangement for the shoe of a roof truss. This is open to the objection that the centre lines of the rafter and tie do not intersect upon the centre line of the column, so that secondary bending stresses will be caused in the sections, apart from those caused by the eccentricity of the rivets.

Fig. 92 (*b*) shows a preferable arrangement, also common, in which the above objection is met, but it has, in common with the form above, another objection that might be avoided, viz., that the rivets will not be equally stressed, as their centroid does not coincide with the imaginary pin-joint.

Fig. 93 shows an arrangement—not common—in which both objections are avoided.

Fig. 94 shows a knee-brace connection to a column. The form shown as (*a*) is quite common, and is open to the objection that the rivets passing through the column, which are in tension as well as shear, are unequally stressed. In the form shown at (*b*) this objection is removed.

Secondary Stresses in Angle-cleat Rivets.—In the case of the rivets in an angle cleat as commonly used for connecting together **I** beams in constructional steelwork, we have an

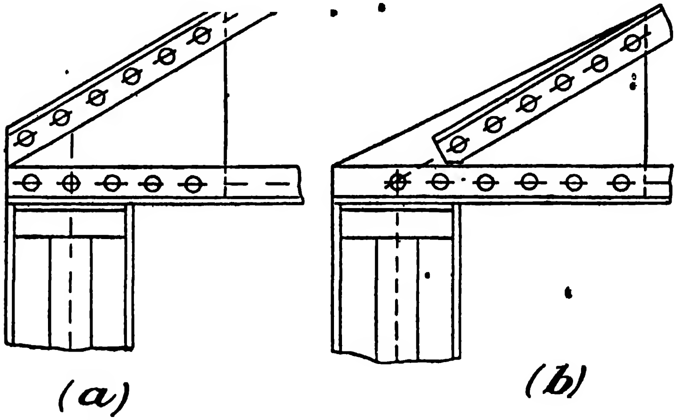


Fig. 92.—Roof Truss Shoes with Secondary Stresses.

interesting example of an eccentrically loaded riveted joint in which secondary stresses arise. The designer does not, of course, calculate the strength of these rivets in the ordinary case, because such connections are to a large extent standardised for a certain minimum length of beam; but in special cases calculations have to be made; and, as the eccentricity of the load causes considerably heavier stresses on the rivets than if the reaction be merely divided between them, it is usual to allow for the eccentricity in the following manner.

We will first consider the general case, and then apply our method to a particular numerical example.

Let 1, 2, 3, 4, 5 . . . Fig. 95, represent a number of rivets

connecting a plate A to a plate B, and let P be the load transmitted from one plate to the other.

Then if X is the centre of gravity of the rivets, and P is at a distance x from X, it follows that we may place at X forces equal to, and opposite to, P, as in the case of an eccentrically loaded

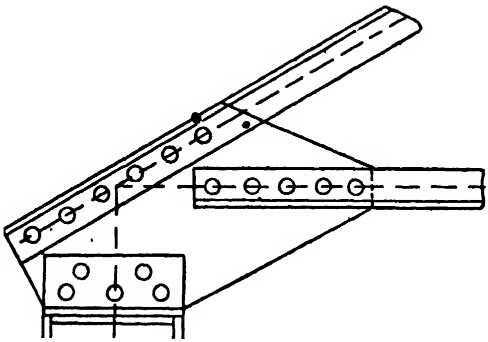


Fig. 93.—Roof Truss Shoe without Secondary Stresses.

column, thus obtaining a couple of moment $M = P\,x$, and a central load P.

The central load P may be divided equally between the rivets, thus giving a force W on each.

The moment $P\,x$ may be divided up between the rivets as follows :—

Let the load on a rivet due to the moment be proportional to the area a of each rivet, and to its distance r from the centre of gravity X.

Say load on each rivet $= k\,a\,r$, where k is a constant,

Then moment of load on each rivet = load × r

$$= k\,a\,r^2 \quad \text{..............................(1)}$$

Total moment of loads on rivets = moment P x

$$= \Sigma\,(k\,a\,r^2) = k\,(a_1\,r_1^{\,2} + a_2\,r_2^{\,2} + \ldots a_n\,r_n^{\,2})$$

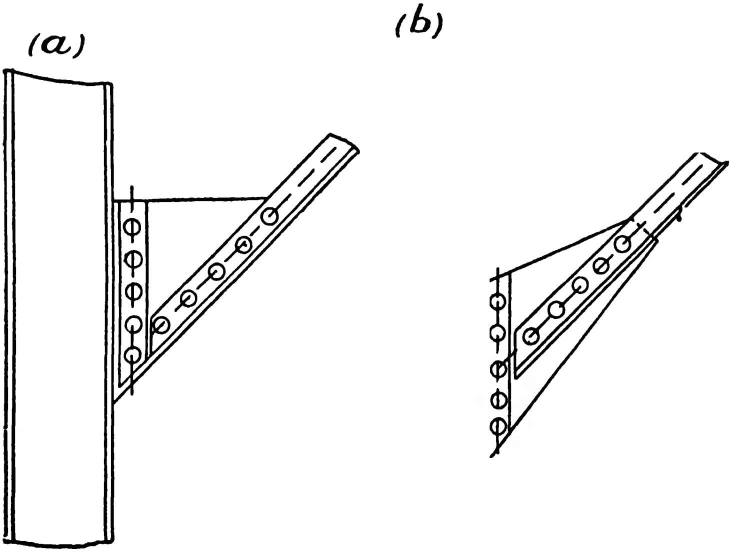


Fig. 94.—Knee-brace with and without Secondary Stresses.

In nearly every case the area of rivets a will be constant, ∴ putting $k\,a = s$ we have

Total moment $= \mathrm{P}\,x = s\,\Sigma\,r^2$ ....(2)

where s = load at unit distance from x.

$$s = \frac{\mathrm{P}\,x}{\Sigma\,r^2}\,* \quad \text{(3)}$$

Then load due to eccentricity on any particular rivet = T $= s\,.\,r$

This will be called the *moment load.*

* It is interesting to compare this formula with that obtained in dealing with continuous portals in the previous chapter (p. 210).

Then the resultant load R on any rivet will be the resultant of the loads T and W, and can be found simply by the usual graphical method. Fig. 95 shows the construction applied to the rivet 3. The resultant of the various loads R should then come equal and opposite to P. This resultant can be found nicely by

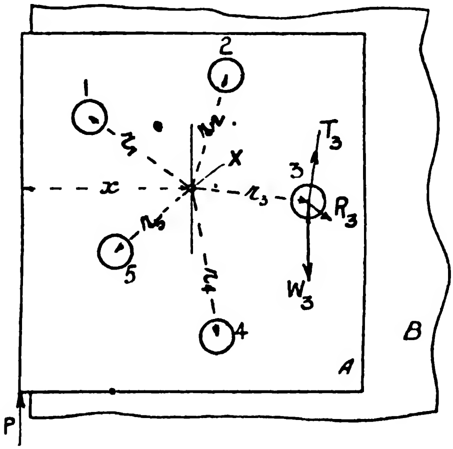


Fig. 95.—Secondary Stresses in Angle-cleat Rivets.

the link and vector polygon construction, and provides an interesting check of the accuracy of the calculations.

NUMERICAL EXAMPLE.—*We will now take the case shown in Fig. 96 of the cleat given in the Handbook of Messrs. Dorman, Long, & Co., Ltd., for a 16 in. by 6 in. standard* **I** *beam with a minimum span of 18 ft., the rivets being of* $\frac{3}{4}$ *in. diameter.*

The safe uniformly distributed load given for this span and beam is 25 tons, so that the reaction at each end will be $\frac{25}{2} = 12{\cdot}5$ tons, and half of this will be carried by each angle, or the load P will be 6·25 tons.

First find the position of the centre of gravity of the rivets. It is clearly on the horizontal line through the rivet 3, and its distance from the line 1, 3, 5 is obtained by moments thus:

$$5\,d = 2 \times 2\tfrac{1}{4}$$

$$\textit{i.e.}\quad d = \frac{4{\cdot}5}{5} = {\cdot}9 \text{ in.}$$

Then we tabulate the dimensions as follows :—

No. of rivet.	r	r^2
1	4·58	21·06
2	2·62	6·88
3	·90	·81
4	2·62	6·88
5	4·58	21·06
		$\Sigma r^2 = 56·69$

$$\therefore \quad s = \frac{P x}{\Sigma r^2} = \frac{6·25 \times 3·15}{56·69}$$
$$= ·348 \text{ tons.}$$

The moment load will be a maximum on rivets 1 and 5 because they are farthest from x, and will be equal to

$$T_5 = ·348 \times 4·58 = 1·59 \text{ tons.}$$

The direct load W on this rivets $= \dfrac{6·25}{5} = 1·25$ tons.

Therefore resultant load $= R_5 = 2·20$ tons. [See Fig. 96.]

Now bearing area of a $\frac{3}{4}$-in. rivet in a $\frac{3}{8}$-in. plate $= \dfrac{3}{4} \times \dfrac{3}{8} = \dfrac{9}{32}$ sq. in.

Bearing stress on rivet $= \dfrac{2·20 \times 32}{9} = 7·82$ tons per sq. in.

Area of a $\frac{3}{4}$-in. rivet in section $= \dfrac{\pi}{4} \times \left(\dfrac{3}{4}\right)^3 = ·442$

$\therefore$ Shear stress on rivet $= \dfrac{2·20}{·442} = 4·98$ tons per sq. in.

The above calculation shows that the rivets are stressed just about up to what is commonly taken as a safe working stress for rivets in shear, viz., 5 tons per sq. in. The importance of allowing for the eccentricity of the stress will be clear from this example, because the resultant maximum stress on the rivets comes nearly twice the value which would have been found if the eccentricity had not been taken into account.

Tests of Cleat Connections.—In the discussion of Mr. Pittman's paper referred to above, Mr. P. S. Whitman gave some very interesting figures of tests on cleat connections.

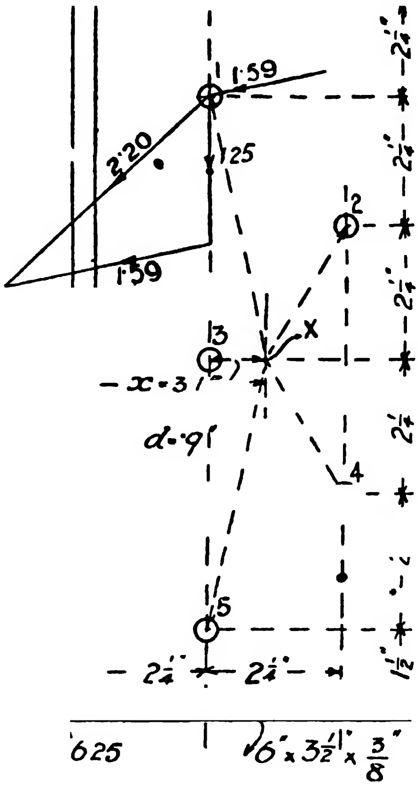


Fig. 96.—Secondary Stresses in Angle-cleat Rivets.

Each cleat had two rivets on one side, for instance, as given by Messrs. Dorman, Long for 6" and 5" beams. The results were rather variable, but in most cases the safe load given by the test was between the values for ordinary primary stress calculations and those allowing for secondary stress.

The following results are interesting :—

Test number.	Safe loads in pounds.		
	By test.	By ordinary method.	By new method.
1	1000	6900	1470
2	6500	6900	1765
3	4450	13,800	2940
4	6250	13,800	3530
5	1000	—	—
6	2000	6900	1470

No. 5 is very interesting; the holes in the beam were $\frac{1}{2}''$ larger in diameter than the rivets, so that the safe load of 1000 lbs. is a measure of the friction exerted by the closing of the rivets; this friction is a very variable quantity in riveted joints and cannot be relied upon with safety for design. Nos. 2 and 4 were riveted in the ordinary way. No. 1 had rough bolts forced in, the nuts being tightened only by hand. Nos. 3 and 6 had turned bolts, with nuts tightened by hand.

It will be noted with interest that in Nos. 3 and 6, which come the nearest to the theoretical conditions of the joints, the results by test agree very much better with the calculations by the new method than by the old.

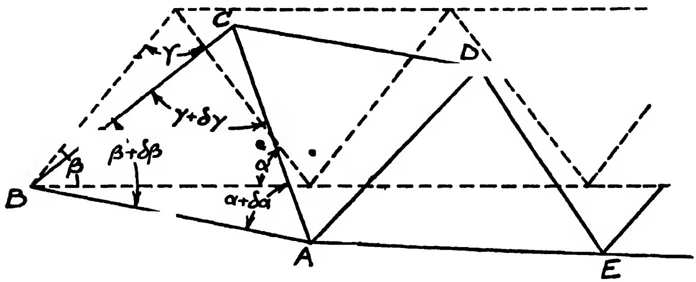
*** Secondary Stresses due to Rigidity of Joints.**— Secondary stresses in framed structures are often caused by the rigidity of joints which are assumed in the ordinary methods of calculation as being freely hinged.

In Fig. 97, (1) shows the deformed form of a Warren girder on exaggerated scale, and (2) shows the bending of the members making up the joint A due to the rigidity of the joint.

By adding the two equations (30) for cross-beams of portals

(p. 182), and putting I for I_B, the bending moment at A in the bar A B due to angular changes θ_{AB} and θ_{BA} is given by

$$M_{AB} = \frac{2\,E\,I}{l}\,(2 \tan \theta_{BA} + \tan \theta_{AB}) \quad \ldots\ldots\ldots\ldots (1)$$



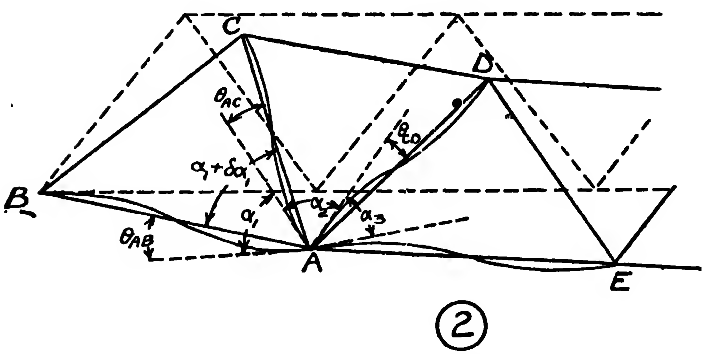


Fig. 97.—Secondary Stresses due to Rigidity of Joints.

or replacing the tangent by the angle, since the latter is very small

$$M_{AB} = \frac{2\,E\,I}{l}\,(2\,\theta_{BA} + \theta_{AB}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now, if the bars meeting at A are concurrent, the sum of the bending moments in the various bars is zero

$$\therefore \Sigma M = M_{AB} + M_{AC} + M_{AD} + M_{AE} = 0 \quad \ldots\ldots\ldots (3)$$

We have first to calculate the changes δa, $\delta \beta$, $\delta \gamma$ in the angles, irrespective of the stiffness of the joint.

In Fig. 98 the triangle A B C is shown in full lines in its original shape, and in dotted lines a deformed shape, caused by

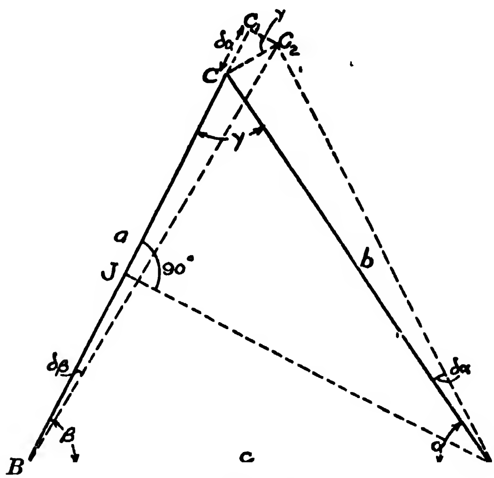


Fig. 98.

stretching the bar *a* by an amount δa, the others being taken as unstrained.

Let f_a, f_b, f_c be the primary stresses in the bars.

$$\text{Then } \delta a = \frac{f_a \cdot a}{E};\ \delta b = \frac{f_b \cdot b}{E};\ \delta c = \frac{f_c \cdot c}{E} \qquad (4)$$

Considering that the displa cements are very small, the angles $C\,C_1\,C_2$ and $A\,C\,C_2$ may be taken as right angles, and therefore the angle $C_1\,C_2\,C = \gamma$.

$$\text{Then } C_1 C_2 = C\,C_1 \cot\gamma = \delta a \cot\gamma \qquad (5)$$

$$\text{Also } \frac{C_1 C_2}{a} = -\delta\beta \text{ [to the first degree of approximation]}$$

$$\therefore \delta \beta = \frac{-\delta a \cot \gamma}{a} = -\frac{f_a}{E} \cot \gamma \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

$\delta \beta$ is taken negative because β is shown as decreasing.

Similarly if A B were changed in length and the other sides were assumed unchanged, we should have

$$\delta \beta = -\frac{f_c}{E} \cot \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

To consider the change in β due to a change in A C, we will note that this corresponds to the change $\delta \alpha$ on the figure in the angle α due to a change of length δa in B C.

$$\text{Now} \quad \delta \alpha = \frac{\text{C C}_2}{b} = \frac{\text{C C}_1}{b \sin \gamma} = \frac{\text{C C}_1}{\text{A J}}$$

$$\text{C J} = \text{A J} \cot \gamma$$

$$\text{And} \quad \text{B J} = \text{A J} \cot \beta$$

$$\therefore \text{C J} + \text{B J} = a = \text{A J} (\cot \beta + \cot \gamma)$$

$$\therefore \text{A J} = \frac{a}{(\cot \beta + \cot \gamma)}$$

$$\therefore \delta \alpha = \frac{\delta a}{\dfrac{a}{(\cot \beta + \cot \gamma)}}$$

$$= \frac{f_a}{E} (\cot \beta + \cot \gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$\therefore$ the corresponding change of β will be given by

$$\delta \beta = \frac{f_b}{E} (\cot \alpha + \cot \gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Adding together the separate changes of angles, we get, when all the sides change in length,

$$\delta \beta = \frac{(f_b - f_a)}{E} \cot \gamma + \frac{(f_b - f_c)}{E} \cot \alpha \quad \ldots\ldots\ldots (10)$$

Similarly for the other angles

$$\delta \alpha = \frac{(f_a - f_c)}{E} \cot \beta + \frac{(f_a - f_b) \cot \gamma}{E} \quad \ldots\ldots\ldots (11)$$

$$\delta \gamma = \frac{(f_c - f_b)}{E} \cot \alpha + \frac{(f_c - f_a) \cot \beta}{E} \quad \ldots\ldots\ldots (12)$$

Determination of Deflection Angles θ at a Joint.—The calculation of the angles θ at any joint can be facilitated, when the

various angles α β, γ, &c., have been found, by noting the following relations,—

$$\left.\begin{aligned}\theta_{AC} &= \theta_{AB} + \delta\,\alpha_1\\ \theta_{CD} &= \theta_{AB} + \delta\,\alpha_1 + \delta\,\alpha_2\\ \theta_{AE} &= \theta_{AB} + \delta\,\alpha_1 + \delta\,\alpha_2 + \delta\,\alpha_3\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(13)$$

Equation (3) can then be written for each joint, expressing the various M's in terms of the θ's as in equation (1), only one θ at each point being an unknown.

We thus derive as many equations as there are joints, and can solve them for the standard θ at each joint, then calculating the separate θ's as in equation (13).

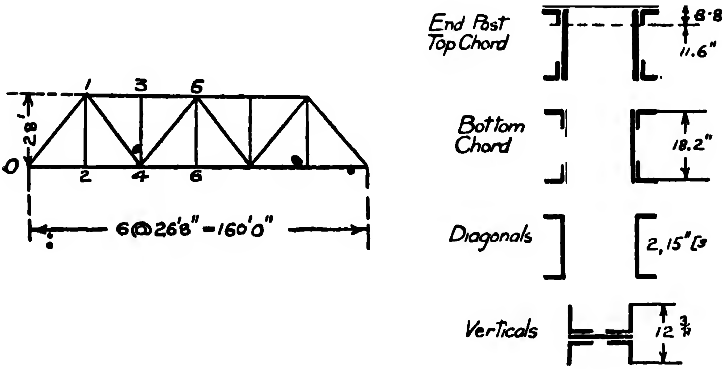
NUMERICAL EXAMPLE.—*Find the secondary stresses due to stiffness of joints in the Pratt Truss shown in Fig. 99, the stresses on the diagram being primary stresses in kips* (1000 *lbs.*) *per sq. in.*

We are indebted to an article by Prof. F. E. Turneare in the *Engineering News* (New York) of September 5th, 1912, for the following example. The necessary data as to primary stresses and sections are given in Table I.

TABLE I.

Member	Length. (in.)	Area A (sq. in.)	Moment of Inertia. I (in.^{4})	Extreme fibre dist. y (in.)	Primary stress S (1000 lbs. or kips).	Unit-stress $s = S/A$ (1000 lbs. or kips per sq. in.)	$K = \frac{I}{l}$
0–1	463	47·7	3280	$\begin{cases}8·8\\11·6\end{cases}$	− 101·0	− 2·12	7·08
1–3	320	47·7	8280	$\begin{cases}8·8\\11·6\end{cases}$	− 112·0	− 2·35	10·25
3–5	320	47·7	3280	$\begin{cases}8·8\\11·6\end{cases}$	− 112·0	− 2·35	10·25
0–2	320	33·0	1410	9·1	+ 69·9	+ 2·12	4·41
2–4	320	33·0	1410	9·1	+ 69·9	+ 2·12	4·41
4–6	320	57·2	1820	9·1	+ 126·0	+ 2·21	5·69
1–4	463	29·4	805	7·5	+ 60·8	+ 2·07	1·74
4–5	463	26·5	750	7·5	− 20·3	− ·77	1·62
1–2	336	19·7	120	6·2	+ 20·0	+ 1·02	0·36
3–4	336	19·7	120	6·2	− 9·3	− ·47	0·36
5–6	336	19·7	120	6·2	+ 20·0	+ 1·02	0·36

Calculations of Changes of Angle, $\delta\, a$.—The data for this calculation are conveniently arranged as shown in Fig. 99. The triangles are lettered A, B, &c., and the primary stresses per unit area (s) are written on the members. Then, proceeding by triangles, and denoting the



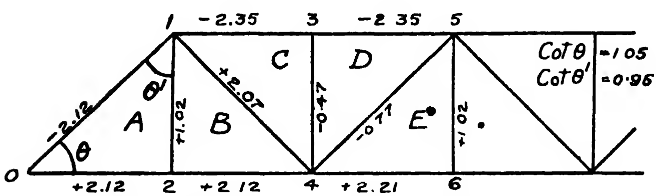


Fig. 99.—*Secondary Stresses in a Pratt Truss.*

angles by the figures at the nodes, we have from equations (10–12) for triangle A, taking E as unity for convenience:

$$\delta\, 0 = (1{\cdot}02 + 2{\cdot}12) \times 1{\cdot}05 = +\ 3{\cdot}30$$

$$\delta\, 1 = (2{\cdot}12 + 2{\cdot}12) \times 0{\cdot}95 = +\ 4{\cdot}03$$

$$\delta\, 2 = (-\ 2{\cdot}12 - 2{\cdot}12) \times 0{\cdot}95 + (-\ 2{\cdot}12 - 1{\cdot}02) \times 1{\cdot}05 = -\ 7{\cdot}3$$

and so on for the other triangles. These results are given in Table II column 4, grouped by joints, as far as joint 6.

Formulation of Equations.—For each joint we now select a particular θ as the reference angle and express all other deflection angles

at that joint in terms of this θ and the changes of angle, as in equation (13). This work is shown in column 5, Table II. At joint 1, for example, the reference angle θ is taken as θ_{13} (called simply θ_1 for convenience), and the various other deflection angles are found by adding in succession the values of $\delta\,\alpha$ given in column 4. For joint 5 the reference angle is taken as θ_{56} and for joint 6, θ_{65}. By reason of symmetry these angles are zero, hence all values of θ for these joints are given directly in column 5.

We now proceed to write out equation (3) for each joint, substituting the values of θ from column 5. In doing this it is convenient to tabulate in column 6 the values of K θ. For joint 0 the equation will be $M_{02} + M_{01} = 0$; or, from (1), omitting the factor 2 E,

$$K\,(2\,\theta_{02} + \theta_{20}) + K\,(2\,\theta_{02} + \theta_{10}) = 0\,;$$

or more conveniently

$$2\,(K\,\theta_{02} + K\,\theta_{01}) + K\,\theta_{20} + K\,\theta_{10} = 0.$$

The parenthesis is given by the sum of all the θ's at joint 0, column 5, and the other values of K θ are found opposite the respective members 2–0 and 1–0.

Combining numerical quantities the equation is

$$22{\cdot}98\,\theta_0 + 7{\cdot}08\,\theta_1 + 4{\cdot}41\,\theta_2 = -\,39{\cdot}2 \;\ldots\ldots\ldots\ldots\ldots\; (a)$$

Equations for joints 1, 2, 3, and 4 are

$$7{\cdot}08\,\theta_0 + 38{\cdot}86\,\theta_1 + 0{\cdot}36\,\theta_2 + 10{\cdot}25\,\theta_3 + \;+\; 1{\cdot}74\,\theta_4 = -\;87{\cdot}0 \;\ldots\; (b)$$

$$4{\cdot}41\,\theta_0 + 0{\cdot}36\,\theta_1 + 18{\cdot}36\,\theta_2 + 4{\cdot}41\,\theta_4 = +\;47{\cdot}0 \;\ldots\; (c)$$

$$10{\cdot}25\,\theta_1 + 41{\cdot}72\,\theta_2 + 0{\cdot}36\,\theta_4 = -\;196{\cdot}7 \ldots\; (d)$$

$$1{\cdot}74\,\theta_1 + 4{\cdot}41\,\theta_2 + 0{\cdot}36\,\theta_3 + 27{\cdot}64\,\theta_4 = +\;86{\cdot}1 \;\ldots\; (e)$$

TABLE II.

Joint	Member	Angle	Change of angle δa	Value of θ in terms of reference θ	K θ	Calculated Value of θ	Bending Moment M 1000 in.-lb. or in.-kips.	Secondary fibre-stress f 1000 lbs. or kips per sq. in.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	0–1			θ_0	$7·08\ \theta_0$	$-1·90$	-45.6	$-0·12$
	0–2	102	$+3·30$	$\theta_0+3·30$	$4·41\ \theta_0+14·6$	$+1·40$	$+45·5$	$+0·29$
1	1–3			θ_1	$10·25\ \theta_1$	$0·83$	$+38.6$	$-0·14$
	1–4	314	$-2·67$	$\theta_1-2·67$	$1·74\ \theta_1-4·64$	$-3·50$	$-18·2$	$+0·17$
	1–2	412	$+0·05$	$\theta_1-2·62$	$0·36\ \theta_1-0·94$	$-3·45$	$-8·6$	$+0·44$
	1–0	210	$+4·03$	$\theta_1+1·41$	$7·08\ \theta^1+10·0$	$+0.58$	$-11·9$	$-0·04$
2	2–0			θ_2	$4·41\ \theta_2$	$+2·35$	$+53·9$	$+0·35$
	2–1	021	$7·33$	$\theta_2-7·33$	$0·36\ \theta_2-2·64$	$-4·98$	$-9·6$	$+0·50$
	2–4	124	$+1·05$	$\theta_2-6·28$	$4·41\ \theta_2-27·7$	$-3·93$	$-44·1$	$+0·28$
3	3–5			θ_3	$10·25\ \theta_3$	$-4·53$	$-121·$	$-0·32$
	3–4	534	$+1·19$	$\theta_3+1·19$	$0·36\ \theta_3+0·43$	$-3·33$	$-6·6$	$-0·34$
	3–1	431	$+6·88$	$\theta_3+8·07$	$10·25\ \theta_3+87·7$	$+3·54$	$+128.$	$-0·34$
4	4–2			θ_4	$4·41\ \theta_4$	$+2·86$	$+15·8$	$+0·10$
	4–1	241	$-1·10$	$\theta_4-1·10$	$1·74\ \theta_4-1·91$	$+1·76$	$+0·06$	$+0·01$
	4–3	143	$-4·21$	$\theta_4-5·31$	$0·36\ \theta_4-1·92$	$-2·45$	-6.0	$-0·31$
	4–5	345	$-1·50$	$\theta_4-6·81$	$1·62\ \theta_4-11·0$	$-3·95$	$-16·4$	$-0·16$
	4–6	546	$+1·88$	$\theta_4-4·93$	$5·69\ \theta_4-28·0$	$-2·07$	$+6·6$	$+0·03$
5	5–6			0	0	0·00	0·00	0·00
	5–4	654	$+2·84$	$+2·84$	$+4·60$	$+2·84$	$+5·6$	$-0·06$
	5–3	453	$+0·31$	$+3·15$	$+32·3$	$+3·15$	$+36·3$	$-0·10$
6	6–5			0	0	0·00	0·00	0·00
	6–4	564	$-4·72$	$+4·72$	$+26·8$	$+4·72$	$+83·8$	$+0·42$

Solution of Equations.—The direct solution of the five equations is not a long process if the elimination is carried out in the proper order. Thus θ_0 occurs in (*a*), (*b*), and (*c*) only, and should first be eliminated, giving two new equations in place of these three. Then θ_2 will be found to occur in these two new equations and in equation (*e*) only. Eliminating θ_2 gives us two new equations containing θ_1, θ_3 and θ_4,

which are then combined with equation (*d*). Generally about five minutes' time is sufficient for each unknown in any single-intersection truss, irrespective of the number of joints or equations, for the number of equations containing any given unknown depends only upon the number of members meeting at the joint in question, being always one more than the number of such members.

The calculated values of θ are as follows : $\theta_0 = -1{\cdot}90$, $\theta_1 = -0{\cdot}83$, $\theta_2 = +2{\cdot}35$, $\theta_3 = -4{\cdot}53$, $\theta_4 = +2{\cdot}86$. These are also given in column 7 of Table II.

The exact method of solution, as employed above seems to be about as expeditious as approximate methods, although errors are not so readily detected. In solving by successive approximations Prof. Turneare suggests the following methods. In the first approximation the values of all coefficients of θ, except the one sought, are assumed equal to zero. Thus, from equation (*a*) for joint o, we get at once

$$\theta_0 = \frac{-39{\cdot}2}{23} = 1{\cdot}7.$$

Likewise from equation (*b*),

$$\theta_1 = \frac{-87{\cdot}0}{38{\cdot}9} = -2{\cdot}2\,;$$

so also $\theta_2 = +2{\cdot}6$, $\theta_3 = -4{\cdot}6$, and $\theta_4 = +3{\cdot}1$. Then for a second approximation use equation (*a*) for θ_0 as before, but substitute the above values for θ_1 and θ_2; and so for the other angles. The second approximations thus obtained are : $\theta_0 = -1{\cdot}5$; $\theta_1 = -0{\cdot}9$, $\theta_2 = +2{\cdot}3$, $\theta_3 = -4{\cdot}2$, and $\theta_4 = +2{\cdot}8$. A third approximation would be sufficient.

Bending Moments and Stresses.—Table II. gives in column 8 the values of the several bending moments calculated from equation (1), and in column 9 the resulting secondary stresses.

Comparison with Experiments.—Within recent years considerable attention has been given to the actual measurements of stresses in structures by means of extensometer readings, this being the only really reliable means of investigating the reliability of theoretical methods. The American Railway Engineering Association has made investigations of this kind upon the secondary stresses in bridge trusses, and their results agree quite well with the results obtained by the above treatment.

INDEX.

London: Printed by Strangeways & Sons, Tower Street, Cambridge Circus, W.C.